Solutions to RSPL/1

1. Given that A is a Skew-symmetric matrix of order 3.

$$\therefore \quad A^T = -A \Rightarrow |A^T| = |-A|$$

$$\Rightarrow |A| = |-1|^3 |A| \qquad (\because \ |A^T| = |A|)$$

$$\Rightarrow |A| = |-A| \Rightarrow 2|A| = 0$$

$$\Rightarrow |A| = 0$$

2. Let $f(x) = x^3 \sin^4 x$

$$\Rightarrow \quad f(-x) = (-x)^3 \sin^4 (-x) = -x^3 \sin^4 x = -f(x)$$

$\Rightarrow$ $f(x)$ is odd function; therefore, by P_7,

$$= \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx = 0$$

3. Given

$$\Rightarrow \quad y = \tan x°$$

$$\Rightarrow \quad y = \tan \frac{\pi x}{180}$$

Differentiate both sides w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \tan \frac{\pi x}{180} = \sec^2 \frac{\pi x}{180} \cdot \frac{\pi}{180} = \frac{\pi}{180} \sec^2 \frac{\pi x}{180}$$

4. Given $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$

$$|\vec{a}| = \sqrt{(2)^2 + (-6)^2 + (3)^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$$

Unit vector in the direction of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{7}(2\hat{i} - 6\hat{j} + 3\hat{k})$

$$= \frac{2}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}$$

5. Given $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$

$$\Rightarrow \quad \begin{bmatrix} x-2 & -10 & 2x-8 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad [x^2 - 2x - 40 + 2x - 8] = [0]$$

$$\Rightarrow \quad x^2 - 48 = 0 \Rightarrow x = \pm\sqrt{48}$$

6. Let θ denote the angle at instant t

Here $$\frac{d\theta}{dt} = 2\frac{d}{dt}\sin\theta$$

$$\Rightarrow \quad \frac{d\theta}{dt} = 2\cos\theta\,\frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{1}{2} = \cos\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$ Hence, the required angle is $\frac{\pi}{3}$.

7. Let $$I = \int \cos^3 x\, e^{\log \sin x}\, dx$$

$$I = \int \cos^3 x \sin x\, dx \qquad (\because \quad e^{\log f(x)} = f(x))$$

Put $\cos x = t \;\Rightarrow\; -\sin x\, dx = dt \;\Rightarrow\; \sin x\, dx = -\,dt$

$$I = -\int t^3\, dt = \frac{-1}{4}t^4 + C$$

$$I = \frac{-1}{4}\cos^4 x + C$$

8. We know that $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{2}{5} + \frac{1}{3} - \frac{1}{5} = \frac{8}{15}$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$= 1 - \frac{8}{15}$$

$$= \frac{7}{15}$$

9. Given $$y = 2\sin x$$

Then $$\frac{dy}{dx} = 2\cos x$$

Also, given that $$x = \frac{\pi}{2} \text{ and } x + \Delta x = \frac{22}{14}$$

Then $$\Delta x = \frac{22}{14} - x = \frac{22}{14} - \frac{\pi}{2}$$

and $$\frac{dy}{dx} = 2\cos x = 2\cos\frac{\pi}{2} = 0$$

Now $$\Delta y \approx \left(\frac{dy}{dx}\right)\Delta x = 0\left(\frac{22}{14} - \frac{\pi}{2}\right) = 0$$

Hence, approximate change in y is 0.

10. Let the person invest ₹ x in bonds of type A and ₹ y in bonds of type B then his earning i.e. return (in ₹)

$$Z = 0.10\,x + 0.15\,y$$

As the person can invest upto ₹ 20000, so investment constraints is

$$x + y \leq 20000$$

Other investment constraints are

$$x \geq 5000,\ y \leq 8000,\ x \geq y$$

Non-negativity constraints are $x \geq 0, y \geq 0$

Thus, mathematical formulation of the L.P.P

Maximize $Z = 0.10\,x + 0.15\,y$

Subject to the constraints

$$x + y \leq 20000,\ x \geq 5000,\ y \leq 8000,\ x \geq y,\ x \geq 0,\ y \geq 0$$

11. (*i*) Given curve is $y = x^3$ ...(*i*)

let the required point on the given curve be (x_1, y_1)

$\therefore$ $y_1 = x_1^3$

Differentiating both sides of (*i*) w.r.t x, we get

$$\frac{dy}{dx} = 3x^2$$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3x_1^2$

Given

Slope of tangent at (x_1, y_1) = y-coordinate of the point (x_1, y_1)

$$3x_1^2 = x_1^3 \Rightarrow 3x_1^2 - x_1^3 = 0 \Rightarrow x_1^2\,(3 - x_1) = 0$$

$\Rightarrow$ $x_1 = 0, 3$

Now the required point is (0, 0) and (3, 27).

12. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = |1| \,.\, \cos\theta = \cos\theta$

Now $(\vec{a} + \vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$

$$= (1)^2 + (1)^2 + 2\cos\theta = 1 + 1 + 2\cos\theta$$

$$= 2 + 2\cos\theta = 2\,(1 + \cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right)$$

Also, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = 4\cos^2\frac{\theta}{2}$

$\Rightarrow$ $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a} + \vec{b}|$

13.

$$\text{L.H.S} = 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

$$= \cos^{-1}\left[\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2}}\right] \qquad \left(\because\ 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$$

$$\text{L.H.S} = \cos^{-1}\left[\frac{(a+b)-(a-b)\tan^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}\right]$$

$$= \cos^{-1}\left[\frac{a\left(1-\tan^2\frac{x}{2}\right)+b\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}\right]$$

$$\text{L.H.S} = \cos^{-1}\left[\frac{\frac{a\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}+b}{a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\right] = \cos^{-1}\left[\frac{a\cos x+b}{a+b\cos x}\right]$$

$\left(\text{Numerator and Denominator divided by } 1+\tan^2\frac{x}{2}\right)$ $\qquad \left(\because\ \cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)$

14. We can represent the given family information as the following matrix:

	Men	Women	Children
F_1	6	5	2
F_2	3	3	4

We can represent the given nutrient requirement information as the following requirement matrix:

	Proteins	Carbohydrates
Men	70	500
Women	50	400
Children	30	300

Then, the matrix of total nutrient requirement of the families can be obtained by multiplying the count matrix and requirement matrix, which is given by

$$\begin{bmatrix}6 & 5 & 2\\ 3 & 3 & 4\end{bmatrix}\begin{bmatrix}70 & 500\\ 50 & 400\\ 30 & 300\end{bmatrix} = \begin{bmatrix}420+250+60 & 3000+2000+600\\ 210+150+120 & 1500+1200+1200\end{bmatrix} = \begin{bmatrix}730 & 5600\\ 480 & 3900\end{bmatrix}$$

Hence, the total requirement of proteins and carbohydrates for family F_1 is 730 g and 5600 g respectively and the total requirement of proteins and carbohydrates for family F_2 is 480 g and 3900 g respectively.

Vitamins, fats and minerals should be included in the diet for a healthy living.

15. Clearly $f(0) = c$

For R.H.L $x > 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{\sqrt{(0+h)+b(0+h)^2} - \sqrt{0+h}}{b(0+h)^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h+bh^2} - \sqrt{h}}{bh^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h(1+bh)} - \sqrt{h}}{b\,h^1 \cdot h^{1/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,\sqrt{1+bh} - \sqrt{h}}{b\,h^1 \cdot h^{1/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,(\sqrt{1+bh} - 1)}{bh\sqrt{h}} \qquad \left(\because\ h^{\frac{1}{2}} = \sqrt{h}\right)$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+bh})^2 - (1)^2}{bh(\sqrt{1+bh}+1)}$$

$$= \lim_{h \to 0} \frac{1+bh-1}{bh(\sqrt{1+bh}+1)}$$

$$= \lim_{h \to 0} \frac{bh}{bh(\sqrt{1+bh}+1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+bh}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

For L.H.L. $x < 0$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{\sin(a+1)(0-h) + \sin(0-h)}{0-h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)(-h) + \sin(-h)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sin(a+1)h}{-h} + \lim_{h \to 0} \frac{-\sin h}{-h} \qquad (\because\ \sin(-\theta) = -\sin\theta)$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h}{(a+1)h}(a+1) + \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (a+1) + 1 \qquad \left(\because\ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right)$$

Since $f(x)$ is continuous at $x = 0$, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\frac{1}{2} = a + 2 = c$$

$$c = \frac{1}{2}, \quad a = \frac{-3}{2} \text{ and } b = R - \{0\}$$

OR

If $$\cos^{-1}\left[\frac{x^2 - y^2}{x^2 + y^2}\right] = \tan^{-1} a$$

$\Rightarrow$ $$\frac{x^2 - y^2}{x^2 + y^2} = \cos(\tan^{-1} a)$$

Differentiate both sides w.r.t x

$\Rightarrow$ $$\frac{d}{dx}\left[\frac{x^2 - y^2}{x^2 + y^2}\right] = \frac{d}{dx}\cos(\tan^{-1} a)$$

$$\frac{(x^2 + y^2)\frac{d}{dx}(x^2 - y^2) - (x^2 - y^2)\frac{d}{dx}(x^2 + y^2)}{(x^2 + y^2)^2} = 0$$

$$\Rightarrow \quad (x^2 + y^2)\left(2x - 2y\frac{dy}{dx}\right) - (x^2 - y^2)\left(2x + 2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) - (x^2 - y^2)\left(x + y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad x^3 - x^2y\frac{dy}{dx} + xy^2 - y^3\frac{dy}{dx} - x^3 - x^2y\frac{dy}{dx} + xy^2 + y^3\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -2x^2y\frac{dy}{dx} + 2xy^2 = 0 \quad \Rightarrow \quad -2x^2y\frac{dy}{dx} = -2xy^2$$

$$\Rightarrow \quad x^2y\frac{dy}{dx} = xy^2 \quad \Rightarrow \quad x\frac{dy}{dx} = y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

16. Let $$I = \int \sin^{-1}\left(\sqrt{\frac{x}{a + x}}\right) dx$$

Put $$x = a\tan^2\theta$$

$$dx = 2a\tan\theta\sec^2\theta\, d\theta$$

$$I = \int \sin^{-1}\left(\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}}\right) 2a\tan\theta\sec^2\theta\, d\theta$$

$$I = \int \sin^{-1}\left(\sqrt{\frac{\tan^2\theta}{1 + \tan^2\theta}}\right) 2a\tan\theta\sec^2\theta\, d\theta$$

$$I = \int \sin^{-1}\left(\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\right) 2a\tan\theta\sec^2\theta\, d\theta$$

$$I = 2a\int \sin^{-1}(\sin\theta)\tan\theta\sec^2\theta\, d\theta \qquad \left(\because \frac{\tan\theta}{\sec\theta} = \sin\theta\right)$$

$$I = 2a\int \underset{I}{\theta}\,\underbrace{\tan\theta\sec^2\theta}_{II}\, d\theta$$

$$I = 2a\left[\theta\frac{\tan^2\theta}{2} - \int 1.\frac{\tan^2\theta}{2}\,d\theta\right] \qquad \text{(Integrate by parts)}$$

$$I = 2a\left[\theta\frac{\tan^2\theta}{2} - \frac{1}{2}\int\tan^2\theta\, d\theta\right] = a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\,d\theta$$

$$I = a\theta\tan^2\theta - a\tan\theta + a\theta + C$$

$$I = a\theta(1 + \tan^2\theta) - a\tan\theta + C$$

$$I = a\tan^{-1}\sqrt{\frac{x}{a}}\left(1 + \frac{x}{a}\right) - a\sqrt{\frac{x}{a}} + C$$

$$I = a\tan^{-1}\sqrt{\frac{x}{a}}\left(\frac{a + x}{a}\right) - \sqrt{ax} + C$$

$$= (a + x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C.$$

17. Let
$$I = \int_{-\pi}^{\pi}\frac{2x(1 + \sin x)}{1 + \cos^2 x}\,dx$$

$$= \int_{-\pi}^{\pi}\frac{2x}{1 + \cos^2 x}\,dx + \int_{-\pi}^{\pi}\frac{2x\sin x}{1 + \cos^2 x}\,dx$$

Here $\frac{2x}{1 + \cos^2 x}$ is an odd function and $\frac{2x\sin x}{1 + \cos^2 x}$ is an even function.

So by the property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0 & \text{if } f(x) \text{ is an odd function} \end{cases}$

$$\int_{-\pi}^{\pi}\frac{2x}{1 + \cos^2 x}\,dx = 0 \text{ and } \int_{-\pi}^{\pi}\frac{2x\sin x}{1 + \cos^2 x}\,dx = 2\int_0^{\pi}\frac{2x\sin x}{1 + \cos^2 x}\,dx$$

$\therefore \qquad I = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \qquad ...(i)$

$$I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx \qquad \left(\text{Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$\Rightarrow \qquad I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2 x}\, dx$

$\Rightarrow \qquad I = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx \qquad ...(ii)$

On adding (i) and (ii), we get

$$2I = 4\int_0^{\pi} \frac{(x + \pi - x)\sin x}{1+\cos^2 x}\, .\, dx$$

$$2I = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Put $\cos x = t \quad \Rightarrow \quad \sin x\, dx = -\, dt$

When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$

$\therefore \qquad 2I = 4\pi \int_1^{-1} \frac{-dt}{1+t^2} = 4\pi \int_{-1}^{1} \frac{dt}{1+t^2} \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$$2I = 4\pi\, [\tan^{-1} t]_{-1}^{1} \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$2I = 4\pi\, [\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$2I = 4\pi \left[\frac{\pi}{4} + \frac{\pi}{4}\right] \qquad \left(\because \tan^{-1}(-1) = -\tan^{-1}(1) = \frac{-\pi}{4}\right)$$

$$2I = 4\pi\left(\frac{\pi}{2}\right)$$

$$I = \pi^2$$

OR

Let $\qquad I = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$

Put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ when $x = 1$, $\quad \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$\left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta \qquad \left(\because \tan\left(\frac{\pi}{4}-A\right)=\frac{1-\tan A}{1+\tan A}\right)$$

$$I = \int_0^{\pi/4} \log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$I = \int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right]d\theta = \int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta$$

$$\left(\because \log\left(\frac{m}{n}\right)=\log m-\log n\right)$$

$$I = \int_0^{\pi/4} \log 2\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$I = \log 2\int_0^{\pi/4} 1\,d\theta - \mathrm{I} \qquad \left(\because I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta\right)$$

$$2I = \log 2\,(\theta)_0^{\pi/4}$$

$$2I = \log 2\left(\frac{\pi}{4}-0\right)$$

$$I = \frac{\pi}{8}\log 2$$

18. Given

$$\cos(x+y)\,dy = dx \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1}{\cos(x+y)}$$

Put $\quad x+y=z \quad \Rightarrow \quad 1+\frac{dy}{dx}=\frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{dz}{dx}-1$

Now

$$\frac{dz}{dx}-1=\frac{1}{\cos z} \quad \Rightarrow \quad \frac{dz}{dx}=\frac{1}{\cos z}+1 \quad \Rightarrow \quad \frac{dz}{dx}=\frac{1+\cos z}{\cos z}$$

$\Rightarrow \quad dz = \dfrac{1+\cos z}{\cos z}\,dx$

$\Rightarrow \quad \dfrac{\cos z}{1+\cos z}\,dz = dx$

On integrating both sides, we get

$\Rightarrow \quad \displaystyle\int \frac{\cos z}{1+\cos z}\,dz = \int dx$

$$\Rightarrow \qquad \int \frac{1+\cos z-1}{1+\cos z}\,dz = \int dx$$

$$\Rightarrow \qquad \int \frac{1+\cos z-1}{1+\cos z}\,dz = \int dx$$

$$\Rightarrow \qquad \int \left(1-\frac{1}{1+\cos z}\right)dz = \int dx$$

$$\Rightarrow \qquad \int 1\,dz - \int \frac{1}{1+\cos z}\,dz = \int dx$$

$$\Rightarrow \qquad z - \int \frac{1}{2\cos^2\frac{z}{2}}\,dz = x + C$$

$$\Rightarrow \qquad z - \frac{1}{2}\int \sec^2\frac{z}{2}\,dz = x + C$$

$$\Rightarrow \qquad z - \frac{1}{2}\,\frac{\tan\frac{z}{2}}{\frac{1}{2}} = x + C$$

$$\Rightarrow \qquad z - \tan\frac{z}{2} = x + C$$

$$\Rightarrow \qquad (x+y) - \tan\left(\frac{x+y}{2}\right) = x + C$$

$$\Rightarrow \qquad x + y - x = \tan\left(\frac{x+y}{2}\right) + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{x+y}{2}\right) + C$$

Given that $y(0) = 0$ i.e. when $x = 0, y = 0$

$$\Rightarrow \qquad 0 = \tan 0 + C \quad \Rightarrow \quad C = 0$$

$\Rightarrow \qquad y = \tan\left(\frac{x+y}{2}\right)$ which is the required particular solution.

19. Given $\vec{a}\times\vec{b} = \vec{c} \Rightarrow \vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ ...(*i*)

Also $\vec{b}\times\vec{c} = \vec{a} \Rightarrow \vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ ...(*ii*)

From (*i*) and (*ii*), it follows that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

Further $\vec{a}\times\vec{b} = \vec{c} \quad \Rightarrow \quad |\vec{a}\times\vec{b}| = |\vec{c}|$

$\Rightarrow \qquad |\vec{a}||\vec{b}|\sin\frac{\pi}{2} = |\vec{c}| \quad \Rightarrow \quad |\vec{a}||\vec{b}| = |\vec{c}|$ ...(*iii*) $(\because \vec{a}\perp\vec{b})$

Also $\qquad \vec{b}\times\vec{c} = \vec{a} \quad \Rightarrow \quad |\vec{b}\times\vec{c}| = |\vec{a}|$

$\Rightarrow \qquad |\vec{b}||\vec{c}|\sin\frac{\pi}{2} = |\vec{a}| \quad \Rightarrow \quad |\vec{b}||\vec{c}| = |\vec{a}|$ $(\because \vec{b}\perp\vec{c})$

$\Rightarrow \qquad |\vec{b}|(|\vec{a}||\vec{b}|) = |\vec{a}|$ (Using (*iii*))

$\Rightarrow \qquad |\vec{b}|^2 |\vec{a}| = |\vec{a}|$

$\Rightarrow \qquad |\vec{b}|^2 = 1$

$\Rightarrow \qquad |\vec{b}| = 1 \qquad (\because |\vec{a}| \neq 0)$

Substituting this value of $|\vec{b}|$ in (*iii*) we get

$$|\vec{a}| \,.\, (1) = |\vec{c}|$$

$$|\vec{a}| = |\vec{c}|$$

20. Here p = probability of A winning a game = 0.4

So, $\qquad q = 1 - p = 0.6$

Now in 'best of 3 games' A wins the match if he wins at least two games out of 3. Its probability $= {}^3C_2 p^2 q + {}^3C_3 p^3 \qquad (\because (q+p)^n = {}^nC_r\,(q)^{n-r} \,.\, p^r)$

$$= 3\,(0.4)^2\,(0.6) + 1\,.\,(0.4)^3$$

$$= (0.4)^2\,(3 \times 0.6 + 0.4)$$

$$= (0.16)\,(2.2) = 0.352$$

Similarly, in 'best of 5 games' A wins the match if he wins at least 3 games out of 5. Its probability $= {}^5C_3\, p^3 q^2 + {}^5C_4\, p^4 q + {}^5C_5\, p^5$

$$= 10\,(0.4)^3\,(0.6)^2 + 5(0.4)^4\,(0.6) + 1 \cdot (0.4)^5$$

$$= (0.4)^3\,[10 \times 0.36 + 5 \times 0.24 + 0.16]$$

$$= (0.064)\,(4.96) = 0.31744$$

Since probability in first case is higher, A should choose 'best of 3 games' match.

21. Equation of any plane passing through the line of intersection of the planes

$$ax + by = 0 \qquad ...(i) \quad \text{and} \quad z = 0 \qquad ...(ii)$$

is $\qquad ax + by + \lambda z = 0 \qquad ...(iii)$

Since the plane (*iii*) makes an angle α with plane (*i*), we have

$$\cos\alpha = \frac{|a\,.\,a + b\,.\,b + 0\,.\,\lambda|}{\sqrt{a^2+b^2}\,\sqrt{a^2+b^2+\lambda^2}}$$

$$= \frac{a^2+b^2}{\sqrt{a^2+b^2}\,\sqrt{a^2+b^2+\lambda^2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+\lambda^2}}$$

$$\Rightarrow \quad \cos^2\alpha = \frac{a^2+b^2}{a^2+b^2+\lambda^2} \quad \Rightarrow \quad \sec^2\alpha = \frac{a^2+b^2+\lambda^2}{a^2+b^2}$$

$$\Rightarrow \quad \sec^2\alpha = 1 + \frac{\lambda^2}{a^2+b^2} \quad \Rightarrow \quad \sec^2\alpha - 1 = \frac{\lambda^2}{a^2+b^2}$$

$$\Rightarrow \quad \tan^2\alpha = \frac{\lambda^2}{a^2+b^2} \quad \Rightarrow \quad \lambda^2 = (a^2+b^2)\tan^2\alpha \Rightarrow \lambda = \pm\left(\sqrt{a^2+b^2}\right)\tan\alpha$$

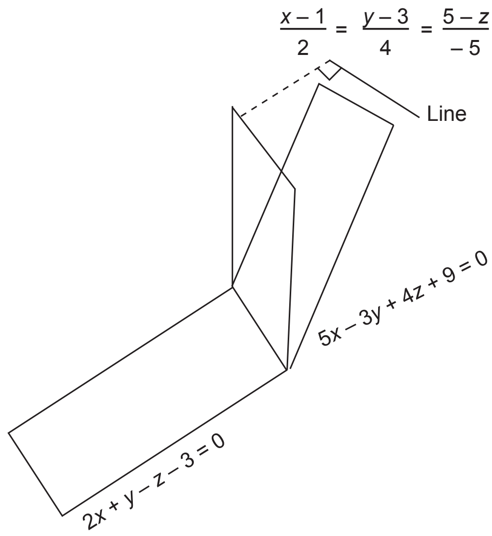
Substituting these values of λ in (*iii*), the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2+b^2}\,\tan\alpha)\,z = 0$.

OR

The given planes are

$$2x + y - z - 3 = 0 \qquad ...(i)$$

$$5x - 3y + 4z + 9 = 0 \qquad ...(ii)$$



The equation of any plane passing through the line of intersection of plane (*i*) and (*ii*) is

$(2x + y - z - 3) + k\,(5x - 3y + 4z + 9) = 0$

i.e.

$(2 + 5k)x + (1 - 3k)y + (-1 + 4k)z - 3 + 9k = 0$ ...(*iii*)

Direction numbers of a normal to plane (*iii*) are $<2 + 5k,\ 1 - 3k,\ -1 + 4k>$

The given line is $\frac{x-1}{2} = \frac{y-3}{4} = \frac{5-z}{-5}$ ...(*iv*)

i.e. $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$

Direction numbers of the line (*iv*) are <2, 4, 5>

Since plane (*iii*) is parallel to line (*iv*) therefore, a normal to plane (*iii*) is perpendicular to line (*iv*)

$\therefore \quad 2(2 + 5k) + 4(1 - 3k) + 5(-1 + 4k) = 0$

$\Rightarrow \quad 4 + 10k + 4 - 12k - 5 + 20k = 0$

$\Rightarrow \quad 3 + 18k = 0 \Rightarrow k = \frac{-1}{6}$

Substituting this value of k in (*iii*), the equation of the required plane is

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(-1 - \frac{4}{6}\right)z - 3 - \frac{9}{6} = 0$$

i.e. $7x + 9y - 10z - 27 = 0.$

22. Let H_1 = letter has come from TATANAGAR

H_2 = letter has come from CALCUTTA

E = two consecutive visible letters are TA

$$P(H_1) = \frac{1}{2} \quad P(H_2) = \frac{1}{2}$$

and $P(E/H_1) = \frac{2}{8}$ and $P(E/H_2) = \frac{1}{7}$

[Since if letter is from TATANAGAR we see that the events of two consecutive letters visible are (TA, AT, TA, AN, NA, AG, GA, AR)]

So $P(E/H_1) = \frac{2}{8}$ and if letter is from CALCUTTA

We see that the events of two consecutive letters visible are

(CA, AL, LC, CU, UT, TT, TA)

So $\qquad P(E/H_2) = \frac{1}{7}$

$$\therefore \qquad P(H_1/E) = \frac{P(H_1)\,P(E/H_1)}{P(H_1)\,P(E/H_1) + P(H_2)\,P(E/H_2)}$$

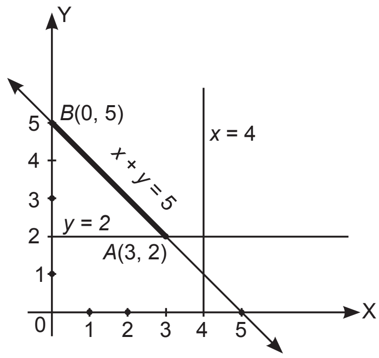
$$= \frac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{2}{8}+\frac{1}{2}\times\frac{1}{7}} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{1}{14}} = \frac{\frac{1}{8}}{\frac{14+8}{14\times 8}}$$

$$= \frac{1}{8}\times\frac{14\times 8}{22} = \frac{14}{22} = \frac{7}{11}.$$

23. Draw the lines

$$x + y = 5, x = 4 \text{ and } y = 2$$

Shade the region satisfied by the given equation $x + y = 5$ and the inequation $x \le 4$, $y \ge 2$, $x \ge 0$.



We observe that the feasible region is only the line segment AB with corner points $A(3, 2)$ and B (0, 5)

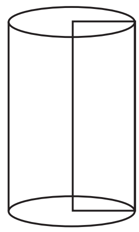
At A (3, 2), $\qquad Z = 5(3) + 8(2) = 31$

at B (0, 5) $\qquad Z = 5(0) + 8(5) = 40$

Thus, Z has minimum value at (3, 2) and the minimum value is 31.

24. Let the radius of cylindrical can be r meter and height be h meter.

Now



$$V = \pi r^2 h$$

$$\Rightarrow \qquad \pi r^2 h = 1 \quad \Rightarrow \quad h = \frac{1}{\pi r^2}$$

let C denote the total cost of can

$$C = (2\pi r h) \times k + (2\pi r^2) \times 2k$$

where k is the cost of construction per unit meter of L.S.A of can

$$C(r) = \left(2\pi r \times \frac{1}{\pi r^2}\right) k + (4\pi r^2) k$$

$$= \frac{2k}{r} + (4\pi r^2)\,k = \frac{2k}{r} + 4k\pi r^2$$

$$C'(r) = \frac{-2k}{r^2} + 8k\pi r$$

In case of maximum and minimum

$$C'(r) = 0$$

$$\Rightarrow \quad \frac{-2k + 8k\pi r^3}{r^2} = 0 \quad \Rightarrow \quad -2k = -8k\pi r^3$$

$$\Rightarrow \quad \frac{2}{8} = \pi r^3 \quad \Rightarrow \quad r = \left(\frac{1}{4\pi}\right)^{\frac{1}{3}}$$

Moreover

$$C''(r) = 4kr^{-3} + 8k\pi = \frac{4k}{r^3} + 8k\pi$$

$$C''\left[\left(\frac{1}{4\pi}\right)^{\frac{1}{3}}\right] = 4k\,(4\pi) + 8k\pi > 0$$

$$24k\pi > 0$$

$\therefore$ Cost is economical

now $$h = \frac{1}{\pi\left(\frac{1}{4\pi}\right)^{2/3}} = \frac{1}{\pi}(4\pi)^{2/3} = \frac{(4\pi)^{2/3}}{\pi} \text{ m}.$$

25. Given that $A = \{1, 2, 3, ..., 9\}$ and $(a, b)\,R\,(c, d)$ if $a + d = b + c$ for $(a, b), (c, d) \in A \times A$

Let $(a, b)\,R\,(a, b) \Rightarrow a + b = b + a\ \forall\ a, b \in A$ which is true for any $a, b \in A$

Hence, R is reflexive

Let $(a, b)\,R\,(c, d) \quad \Rightarrow \quad a + d = b + c$

$$\Rightarrow \quad b + c = a + d \Rightarrow c + b = d + a \Rightarrow (c, d)\,R\,(a, b)$$

So R is symmetric

Let $(a, b)\,R\,(c, d)$ and $(c, d)\,R\,(e, f)$

$$\Rightarrow \quad a + d = b + c \text{ and } c + f = d + e$$

$$\Rightarrow \quad a + d = b + c \text{ and } d + e = c + f$$

$$\Rightarrow \quad (a + d) - (d + e) = (b + c) - (c + f)$$

$$\Rightarrow \quad a - e = b - f \Rightarrow a + f = b + e$$

$(a, b)\,R\,(e, f)$ So, R is transitive

Hence R is an equivalence relation.

Now equivalence class containing

$[(2, 5)]$ is $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ $(\because a + d = b + c, a + 5 = b + 2, a = b - 3)$

OR

We have $f = \{(1, a), (2, b), (3, c)\}$, $g = \{(a, \text{apple}), (b, \text{ball}), (c, \text{cat})\}$.

Now $\quad gof(x) = g(f(x))$

$$\left.\begin{aligned} gof(1) &= g(f(1)) = g(a) = \text{apple} \\ gof(2) &= g(f(2)) = g(b) = \text{ball} \\ gof(3) &= g(f(3)) = g(c) = \text{cat} \end{aligned}\right\} \quad gof = \{(1, \text{apple}), (2, \text{ball}), (3, \text{cat})\}$$

Now $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$ $g^{-1} = \{(\text{apple}, a), (\text{ball}, b), (\text{cat}, c)\}$

$$(gof)^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$$

$$f^{-1}og^{-1}(x) = f^{-1}(g^{-1}(x))$$

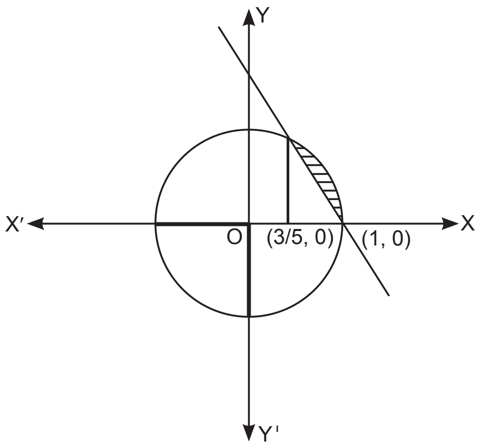
$$\left.\begin{aligned} f^{-1}og^{-1}(\text{apple}) &= f^{-1}(g^{-1}(\text{apple})) = f^{-1}(a) = 1 \\ f^{-1}og^{-1}(\text{ball}) &= f^{-1}(g^{-1}(\text{ball})) = f^{-1}(b) = 2 \\ f^{-1}og^{-1}(\text{cat}) &= f^{-1}(g^{-1}(\text{cat})) = f^{-1}(c) = 3 \end{aligned}\right\}$$

$$f^{-1}og^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\} = (gof)^{-1}$$

26. The equation

$$x^2 + y^2 = 1 \qquad ...(i)$$

represents a circle with centre (0, 0) and radius 1 unit.



The equation $x + \frac{y}{2} = 1$ i.e. $y = 2 - 2x$ represents a straight line.

The circle and line meet where

$$x^2 + (2 - 2x)^2 = 1$$

$$\Rightarrow \qquad 5x^2 - 8x + 3 = 0 \Rightarrow x = 1, \frac{3}{5}$$

$$\text{Required area} = \int_{3/5}^{1} [\sqrt{1 - x^2} - (2 - 2x)]\,dx$$

$$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x - 2x + x^2\right]_{3/5}^{1}$$

$$= \left(0 + \frac{1}{2}\sin^{-1}1 - 2 + 1\right) - \left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} + \frac{1}{2}\sin^{-1}\frac{3}{5} - 2\cdot\frac{3}{5} + \frac{9}{25}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{6}{25} - \frac{1}{2}\sin^{-1}\frac{3}{5} + \frac{6}{5} - \frac{9}{25}$$

$$= \left(\frac{\pi}{4} - \frac{2}{5} - \frac{1}{2}\sin^{-1}\frac{3}{5}\right) \text{ sq. units}$$

Note: The area of the region to be calculated is $\{(x, y): x^2 + y^2 \le 1 \le x + \frac{y}{2}\}$

OR

The given parabola is $y^2 = x$. ...(*i*)

It represents a right hand parabola with vertex at (0, 0)

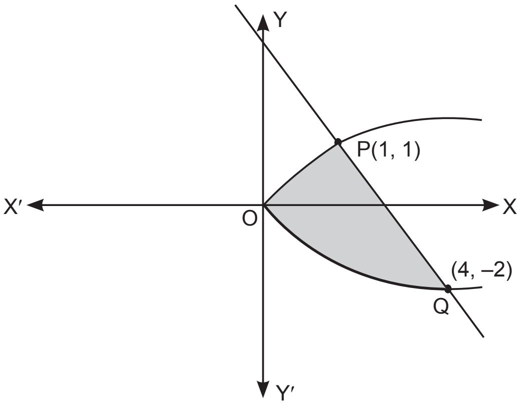
The given line is $y + x = 2$ ...(*ii*)

Solving (*i*) and (*ii*), we get

$$y^2 = 2 - y \Rightarrow y^2 + y - 2 = 0$$

$\Rightarrow \qquad (y-1)(y+2) = 0 \Rightarrow y = 1, -2$

The required area = area of the shaded region



$$= \int_{-2}^{1} [(2-y) - y^2]\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3} = 4\frac{1}{2} \text{ sq. units}$$

27. Let α, β and γ be the angles made by $\vec{n}$ with x, y and z axis respectively. It is given that $\alpha = \beta = \gamma$

$\Rightarrow \qquad \cos\alpha = \cos\beta = \cos\gamma \Rightarrow l = m = n$

where l, m, n are direction cosine of $\vec{n}$

But $\qquad l^2 + m^2 + n^2 = 1$

$\Rightarrow \qquad 3l^2 = 1 \Rightarrow l = \dfrac{1}{\sqrt{3}}$

Thus $\qquad \vec{n} = 2\sqrt{3}\left(\dfrac{1}{\sqrt{3}}\hat{i} + \dfrac{1}{\sqrt{3}}\hat{j} + \dfrac{1}{\sqrt{3}}\hat{k}\right) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$$\left[\because \frac{\vec{n}}{|\vec{n}|} = \text{d.c's vector} = l\hat{i} + m\hat{j} + n\hat{k} \qquad \vec{n} = |\vec{n}|(l\hat{i} + m\hat{j} + n\hat{k})\right]$$

Thus required plane passes through the point (1, –1, 2) and direction of plane is $\vec{n} = 2\hat{i} + 2\hat{j} + 2\hat{k}$.

Now DR's of plane <2, 2, 2> and passing point (1, –1, 2) cartesian equation of plane

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow \qquad 2(x-1) + 2(y+1) + 2(z-2) = 0$

$\Rightarrow \qquad 2x + 2y + 2z - 4 = 0$

$\Rightarrow \qquad x + y + z - 2 = 0$

Vector equation of plane

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$$

28. Given $\qquad \sqrt{1 + x^2 + y^2 + x^2y^2} + xy\dfrac{dy}{dx} = 0$

By simplifying the equation, we get

$$xy\frac{dy}{dx} = -\sqrt{1 + x^2 + y^2 + x^2y^2}$$

$$\Rightarrow \quad xy\frac{dy}{dx} = -\sqrt{(1+x^2)+y^2(1+x^2)}$$

$$= -\sqrt{(1+x^2)(1+y^2)} = -\sqrt{(1+x^2)}\sqrt{(1+y^2)}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}}dy = \frac{-\sqrt{1+x^2}}{x}dx$$

Integrating both sides we get

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{\sqrt{1+x^2}}{x}dx \qquad ...(i)$$

Let $\quad 1 + y^2 = t \Rightarrow 2ydy = dt$ (For LHS)

and $\quad 1 + x^2 = m^2 \Rightarrow 2xdx = 2mdm \Rightarrow xdx = mdm$

$$\Rightarrow \quad dx = \frac{m}{x}dm \qquad \text{(For RHS)}$$

$$\therefore \ (i) \quad \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\int \frac{m}{m^2-1}\cdot mdm \qquad (\because\ x^2 = m^2 - 1)$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\int \frac{m^2}{m^2-1}dm$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} = -\int \frac{m^2-1+1}{m^2-1}dm$$

$$\Rightarrow \quad t^{1/2} = -\int \frac{m^2-1}{m^2-1}dm - \int \frac{dm}{m^2-1}$$

$$\Rightarrow \quad t^{1/2} = -\int 1\,dm - \frac{1}{2(1)}\log\left|\frac{m-1}{m+1}\right| + C$$

$$\left(\because\ \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right)$$

$$t^{1/2} = -m - \frac{1}{2}\log\left|\frac{m-1}{m+1}\right| + C$$

$$\Rightarrow \quad \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$$

29. Let x, y, z be amount of prize to be awarded in the field of agriculture, education and social service respectively. The given situation can be written in matrix form as $AX = B$

Where $\quad A = \begin{bmatrix} 10 & 5 & 15 \\ 15 & 10 & 5 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 70000 \\ 55000 \\ 6000 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 10 & 5 & 15 \\ 15 & 10 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 10(10-5) - 5(15-5) + 15(15-10) = 75 \neq 0$

Hence, A^{-1} exists and system have unique solution

Now $$\text{Adj } A = \begin{bmatrix} 5 & 10 & -125 \\ -10 & -5 & 175 \\ 5 & -5 & 25 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{15}\begin{bmatrix} 1 & 2 & -25 \\ -2 & -1 & 35 \\ 1 & -1 & 5 \end{bmatrix}$$

Putting the value of X, A^{-1}, B in $X = A^{-1}B$, we get

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 1 & 2 & -25 \\ -2 & -1 & 35 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 70000 \\ 55000 \\ 6000 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 70000 + 110000 - 150000 \\ -140000 - 55000 + 210000 \\ 70000 - 55000 + 30000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 30000 \\ 15000 \\ 45000 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$$\Rightarrow x = 2000, y = 1000, z = 3000$$

OR

Given $$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$$

Now, $$\text{L.H.S} = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = \begin{vmatrix} 2x & a+x & a+p \\ 2y & b+y & b+q \\ 2z & c+z & c+r \end{vmatrix}$$

[On applying $C_1 \to C_1 + C_2 - C_3$]

$$= 2\begin{vmatrix} x & a+x & a+p \\ y & b+y & b+q \\ z & c+z & c+r \end{vmatrix}$$ [On taking 2 common from C_1]

$$= 2\begin{vmatrix} x & a & a+p \\ y & b & b+q \\ z & c & c+r \end{vmatrix}$$ [On applying $C_2 \to C_2 - C_1$]

$$= 2\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$$ [On applying $C_3 \to C_3 - C_2$]

$$= +2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$ [On applying $C_1 \leftrightarrow C_2$ and $C_2 \leftrightarrow C_3$]

$= 2(16) = 32$