Solutions to RSPL/3

1. We know that

$$|k\vec{a}| = |k||\vec{a}| = |k| \cdot 3 = 3|k|$$

Now, $-2 \le k \le 1 \Rightarrow 0 \le |k| \le 2$

$\Rightarrow$ $0 \le 3|k| \le 6 \Rightarrow 0 \le |k\vec{a}| \le 6$

2. Given $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Then, $|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

Hence, for any natural number n, we have determinant $(A^n) = |A^n| = |A|^n = 1$

3. Given $A = \{1, 2\}$

Then, $A \times A = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Since, $R = \{(a, b): a + b > 0\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\} = A \times A$

Hence, R is a universal relation on set A.

4. Let e be the identity element in R for the binary operation * on R, then $a * e = e * a = a$ for all $a \in R$.

$\Rightarrow$ $\frac{4ae}{9} = a$ and $e = \frac{9}{4}$

Hence, $\frac{9}{4}$ is the identity element in R.

5. Given $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Putting $a = \tan\alpha$ and $x = \tan\beta$ we get

$$\sin^{-1}(\sin 2\alpha) + \cos^{-1}(\cos 2\alpha) = \tan^{-1}(\tan 2\beta)$$

$\Rightarrow$ $2\alpha + 2\alpha = 2\beta \Rightarrow 2\alpha = \beta$

$\Rightarrow$ $2\tan^{-1} a = \tan^{-1} x$

$\Rightarrow$ $\tan^{-1} a + \tan^{-1} a = \tan^{-1} x$

$\Rightarrow$ $\tan^{-1}\left[\frac{a+a}{1-a^2}\right] = \tan^{-1} x$ $\quad [\because (a)(a) < 1 \text{ as } a \in (0, 1)]$

$\Rightarrow$ $\frac{2a}{1-a^2} = x$

Hence, $x = \frac{2a}{1-a^2}$ is the required solution.

6. Given $A^2 - 4A + I = 0$

Pre-multiplying both sides by A^{-1} we get

$\Rightarrow$ $A^{-1}A^2 - 4A^{-1}A + A^{-1}I = 0$

$\Rightarrow \quad (A^{-1}A)A - 4I + A^{-1} = 0 \qquad (\because A^{-1}A = I, IA^{-1} = A^{-1}, AI = IA = A)$

$\Rightarrow \quad IA - 4I + A^{-1} = 0$

$\Rightarrow \quad A - 4I + A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

7. Given $\quad y = \sin^{-1}\left(x\sqrt{1-x^4} - x^2\sqrt{1-x^2}\right)$

Putting $x^2 = \sin\alpha$ and $x = \sin\beta$, we get

$$y = \sin^{-1}\left(\sin\beta\sqrt{1-\sin^2\alpha} - \sin\alpha\sqrt{1-\sin^2\beta}\right)$$

$$y = \sin^{-1}(\cos\alpha\sin\beta - \sin\alpha\cos\beta)$$

$$= \sin^{-1}(\sin\beta\cos\alpha - \sin\alpha\cos\beta)$$

$$y = \sin^{-1}[\sin(\beta-\alpha)] = \beta - \alpha \qquad (\because \sin^{-1}(\sin x) = x)$$

$$y = \sin^{-1}x - \sin^{-1}x^2$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}\sin^{-1}x^2$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(x^2)^2}}\cdot 2x$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^4}}$$

8. Let a denote the side and A denote the area of the equilateral triangle at instant t.

Here $\quad \frac{da}{dt} = 2$ cm/s; $\frac{dA}{dt} = ?$

Now $\quad A = \frac{\sqrt{3}}{4}a^2$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4}(2a)\frac{da}{dt} = \frac{\sqrt{3}}{4}2(a)\cdot 2 = \sqrt{3}\,a$$

When $a = 20$, we have $\frac{dA}{dt} = 20\sqrt{3}$ cm²/s

Hence, required rate of change $= 20\sqrt{3}$ cm²/s

9. Let $\quad I = \int[\sin(\log x) + \cos(\log x)]dx \qquad$ Put $\log x = t \Rightarrow x = e^t$

$dx = e^t\,dt$

$$I = \int(\sin t + \cos t)\,e^t\,dt$$

$$I = \int \underset{\text{II}}{e^t}\,\underset{\text{I}}{\sin t}\,dt + \int e^t\cos t\,dt$$

$$I = \sin t\int e^t\,dt - \int\left(\frac{d}{dt}(\sin t)\int e^t\,dt\right)dt + \int e^t\cos t\,dt$$

$$I = \sin t\,e^t - \int\cos t\,e^t\,dt + \int e^t\cos t\,dt$$

$$I = e^t\sin t + C = x\sin(\log x) + C$$

10. Given $$e^y(1 + x^2)dy - \frac{x}{y}dx = 0$$

$$\Rightarrow \quad ye^y\,dy = \frac{x}{1+x^2}dx$$

On integrating both sides, we get

$$\int \underset{\text{I}}{y}\,\underset{\text{II}}{e^y}\,dy = \int \frac{x}{1+x^2}dx$$

$$\Rightarrow \quad y\int e^y\,dy - \int\left(\frac{d}{dy}(y)\int e^y\,dy\right)dy = \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$\Rightarrow \quad ye^y - \int e^y\,dy = \frac{1}{2}\log|1+x^2| + C$$

$$\Rightarrow \quad ye^y - e^y = \frac{1}{2}\log|1+x^2| + C$$

$$\Rightarrow \quad (y-1)e^y = \frac{1}{2}\log|1+x^2| + C$$

11. Given $\vec{a} = \alpha\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + \beta\hat{k}$. Then vectors $\vec{a}$ and $\vec{b}$ will be collinear if $\vec{a} = \lambda\vec{b}$, for some (non-zero) scalar λ.

$$\Rightarrow \quad \alpha\hat{i} + 3\hat{j} - 6\hat{k} = \lambda(2\hat{i} - \hat{j} + \beta\hat{k})$$

$$\Rightarrow \quad \alpha = 2\lambda,\ 3 = -\lambda,\ -6 = \lambda\beta$$

$$\Rightarrow \quad \alpha = 2(-3),\ -6 = (-3)\beta$$

$$\Rightarrow \quad \alpha = -6,\ \beta = 2$$

Hence, the given vectors are collinear if $\alpha = -6$, $\beta = 2$.

12. Let S be the sample space associated with the random experiment of throwing a die 3 times, then S has $6 \times 6 \times 6$, i.e. 216 equally likely elementary events (out comes). Let A, B be the events defined as:

A = getting 15 as the sum in a throw of dice three times

B = getting 4 on the first throw

Then B has $1 \times 6 \times 6$, i.e. 36 equally outcomes, out of which only two outcomes (4, 5, 6) and (4, 6, 5) are favourable to A.

$$\therefore \quad \text{Required probability} = P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2}{36} = \frac{1}{18}$$

13. Total amount received by trust = ₹ 30000

Amount spent for medical and educational care (MEC) of the children = ₹ 15000

Amount deposits in a private bank (30000 – 15000) = ₹ 15000

The rate of interest received from medical and educational care provides = 2%

Let the rate of interest received from the bank = x%

We can represent the given expenditure information as the following Expenditure Matrix:

	MEC	Bank
Expenditure	[15000	15000]

We can represent the given interest information as the following Interest Matrix:

$$\begin{array}{cc} & \text{Interest} \\ \text{MEC} & \begin{bmatrix} 2\% \\ \\ x\% \end{bmatrix} \\ \text{Bank} & \end{array}$$

Then, the matrix of total earning can be obtained by multiplying the expenditure matrix and interest matrix, which is given by

$$[15000 \quad 15000]\begin{bmatrix} 2\% \\ x\% \end{bmatrix} = \left[\frac{2}{100}\times 15000 + \frac{x}{100}\times 15000\right] = [300 + 150x] \qquad ...(i)$$

Given that total earning per month = ₹ 1800

We can represent the given earning information as the following Earning Matrix:

$$\text{Earning } [1800] \qquad ...(ii)$$

From (*i*) and (*ii*), we get

$$[300 + 150x] = [1800]$$

$$\Rightarrow \quad 300 + 150x = 1800$$

$$\Rightarrow \quad x = 10$$

Hence, the rate of interest received from the bank = 10%

Yes, people should donate to trust which take care of handicapped children.

14. Given $$y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{1}{1+\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right)^2} \cdot \sqrt{\frac{a-b}{a+b}} \sec^2 \frac{x}{2} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{\sqrt{a-b}}{\sqrt{a+b}\sqrt{a^2-b^2}} \cdot \frac{(a+b)\sec^2 \frac{x}{2}}{(a+b)+(a-b)\tan^2 \frac{x}{2}}$$

$$= \frac{1}{(a+b)\cos^2 \frac{x}{2} + (a-b)\sin^2 \frac{x}{2}}$$

$$\frac{dy}{dx} = \frac{1}{a\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) + b\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)}$$

$$= \frac{1}{a + b\cos x} \qquad \left(\begin{array}{l} \because \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \\ \cos^2\theta + \sin^2\theta = 1 \end{array}\right)$$

OR

Given $f(x) = \begin{cases} x^2 & x \le c \\ ax + b & x > c \end{cases}$

and $f(c) = c^2$

Since f is differentiable at $x = c$, it must be continuous at $x = c$

$$\Rightarrow \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

$$\Rightarrow \quad \lim_{x \to c^-} x^2 = \lim_{x \to c^+} (ax + b) = c^2$$

$$\Rightarrow \quad c^2 = ac + b = c^2$$

$$\Rightarrow \quad c^2 = ac + b \qquad ...(i)$$

As f is differentiable at $x = c$

$$\Rightarrow \quad Rf'(c) = Lf'(c)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{a(c+h) + b - c^2}{h} = \lim_{h \to 0} \frac{(c-h)^2 - c^2}{-h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{ac + ah + b - c^2}{h} = \lim_{h \to 0} \frac{c^2 + h^2 - 2ch - c^2}{-h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{(ac + b) + ah - c^2}{h} = \lim_{h \to 0} \frac{h(h - 2c)}{-h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{c^2 + ah - c^2}{h} = \lim_{h \to 0} \frac{(h - 2c)}{-1} \qquad (\because \ ac + b = c^2)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{ah}{h} = +2c$$

$$\Rightarrow \quad a = +2c$$

$$\Rightarrow \quad 2c = a$$

From (i), we get $\quad c^2 = 2c^2 + b \Rightarrow b = -c^2$

Hence, $a = 2c$ and $b = -c^2$

15. Let $I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Put $x = a \sin \theta \Rightarrow dx = a \cos \theta \, d\theta$

when $x = 0 \quad \theta = 0$ and when $x = a \quad \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{a \cos \theta}{a \sin \theta + \sqrt{a^2 - a^2 \sin^2 \theta}} \, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{a \cos \theta}{a \sin \theta + a \cos \theta} \, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} \, d\theta \qquad ...(i)$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta \, d\theta}{\cos \theta + \sin \theta} \qquad ...(ii)$$

On adding (i) and (ii) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sin\theta + \cos\theta} + \int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin\theta + \cos\theta) d\theta}{(\sin\theta + \cos\theta)} = \int_0^{\frac{\pi}{2}} d\theta$$

$$2I = \left[\theta\right]_0^{\frac{\pi}{2}} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

OR

Let $$I = \int_0^2 |x^2 + 2x - 3|\, dx = \int_0^2 |(x-1)(x+3)|\, dx$$

$\Rightarrow$ $$|x^2 + 2x - 3| = \begin{cases} -(x^2 + 2x - 3) & 0 \le x < 1 \\ x^2 + 2x - 3 & 1 \le x < 2 \end{cases}$$

$\therefore$ $$\int_0^2 |x^2 + 2x - 3|\, dx = \int_0^1 -(x^2 + 2x - 3)\, dx + \int_1^2 (x^2 + 2x - 3)\, dx$$

$$= -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$$= -\left[\left(\frac{1}{3} + 1 - 3\right) - 0\right] + \left[\left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)\right]$$

$$= -\left(-\frac{5}{3}\right) + \frac{2}{3} - \left(-\frac{5}{3}\right) = \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4$$

16. We have, $\sqrt{x} + \sqrt{y} = 4$ ...(i)

$\Rightarrow$ $$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4$$

Differentiating w.r.t. x, we get

$\Rightarrow$ $$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Since, tangent is equally inclined to the axes

$$\frac{dy}{dx} = \pm 1 \Rightarrow \quad -\frac{\sqrt{y}}{\sqrt{x}} = \pm 1$$

S.B.S.

$\Rightarrow$ $$\frac{y}{x} = 1 \quad \Rightarrow \quad y = x$$

From (i) we get $\sqrt{y} + \sqrt{y} = 4 \quad \Rightarrow \quad 2\sqrt{y} = 4$

$\Rightarrow$ $$4y = 16 \ \Rightarrow \ y = 4$$

So, the required co-ordinates are (4, 4).

OR

Given $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x), D_f = R$

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x}\left[\frac{2x}{2\sqrt{1+x^2}} - 1\right]$$

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x} \cdot \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

Since $x^2 \geq 0$ for all $x \in R$ so $1 + x^2 \geq 1$ and $\sqrt{1+x^2} \geq 1$

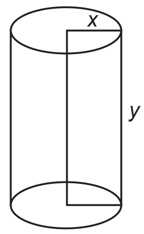
$\Rightarrow$ $$\frac{1}{1+x^2} \leq 1 \text{ and } \frac{1}{\sqrt{1+x^2}} \leq 1$$

$\Rightarrow$ $$2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$$

$\Rightarrow$ $f'(x) \geq 0$ for all $x \in R$

$\Rightarrow$ $f(x)$ is increasing in R.

17. Let x cm and y cm be the dimensions of the rectangle, then its



perimeter $= 2x + 2y = 36$ (given)

$\Rightarrow$ $y = 18 - x$ ...(i)

When the rectangle is revolved about side of length y cm, then it sweeps out a right circular cylinder of radius x cm and height y cm. If v cm^3 is the volume of cylinder,

then $v = \pi x^2 y = \pi x^2(18 - x)$ (Using (i))

$\Rightarrow$ $v = \pi(18x^2 - x^3)$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \pi(36x - 3x^2)$$

$\Rightarrow$ $$\frac{dv}{dx} = 0$$

$\Rightarrow$ $\pi(36x - 3x^2) = 0$

$\Rightarrow$ $x = 12$ $(\because\ x > 0)$

Now, $$\frac{d^2v}{dx^2} = \pi(36 - 6x)$$

$$\left(\frac{d^2v}{dx^2}\right)_{x=12} = \pi[36 - 6(12)] = \pi(36 - 72) = -36\pi < 0$$

$\Rightarrow$ v is maximum when $x = 12$

when $x = 12, y = 18 - 12 = 6$

Maximum volume (in cm^3) $= \pi(12)^2\,(18 - 12) = \pi(12)^2\,(6) = 864\pi$ cm^3

18. Let
$$I = \int (x-2)\sqrt{\frac{x+3}{x-3}}\,dx$$
$$I = \int (x-2)\sqrt{\frac{(x+3)(x+3)}{(x-3)(x+3)}}\,dx$$
$$= \int (x-2)\sqrt{\frac{(x+3)^2}{x^2-9}}\,dx$$
$$I = \int \frac{(x-2)(x+3)}{\sqrt{x^2-9}}dx = \int \frac{x^2+x-6}{\sqrt{x^2-9}}dx = \int \frac{(x^2-9)+x+3}{\sqrt{x^2-9}}dx$$
$$I = \int \frac{x^2-9}{\sqrt{x^2-9}}dx + \int \frac{x+3}{\sqrt{x^2-9}}dx$$
$$I = \int \sqrt{x^2-9}\,dx + \int \frac{x}{\sqrt{x^2-9}}dx + 3\int \frac{dx}{\sqrt{x^2-(3)^2}}$$
$$I = \frac{1}{2}\left[x\sqrt{x^2-9} - \frac{9}{2}\log\left|x+\sqrt{x^2-9}\right|\right] + \frac{1}{2}\cdot 2\sqrt{x^2-9} + 3\log\left|x+\sqrt{x^2-9}\right| + C$$
$$\left(\because \int \frac{x}{\sqrt{x^2-9}}dx = 2\sqrt{x^2-9}+C \quad \text{Use substitution method}\right)$$

19. Given
$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(y^2+y+1)}{x^2+x+1}$$
$$\Rightarrow \quad \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$
$$\Rightarrow \quad \frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$$
$$\Rightarrow \quad \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 0$$

On integrating both sides, we get
$$\Rightarrow \quad \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = C$$
$$\Rightarrow \quad \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = C$$
$$\Rightarrow \quad \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}C \qquad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right)$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right] = \frac{\sqrt{3}}{2}C \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right)$$

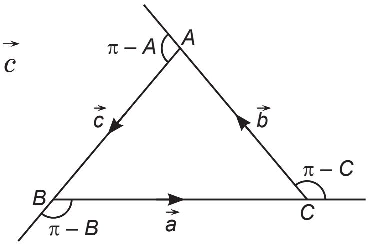
$$\Rightarrow \quad \frac{\sqrt{3}(2y+2x+2)}{3-(4xy+2y+2x+1)} = \tan\left(\frac{\sqrt{3}}{2}C\right)$$

$$\Rightarrow \quad \frac{2\sqrt{3}(x+y+1)}{2(1-x-y-2xy)} = \tan\left(\frac{\sqrt{3}}{2}C\right)$$

$$\Rightarrow \quad \frac{x+y+1}{1-x-y-2xy} = \frac{1}{\sqrt{3}}\tan\left(\frac{\sqrt{3}}{2}C\right) = A \text{ (say)}$$

$\Rightarrow$ $x + y + 1 = A(1 - x - y - 2xy)$ where A is arbitrary constant.

20. In any triangle ABC,



Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

From ΔABC, by triangle law of vectors, we get

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{0} \quad \Rightarrow \quad \vec{b} + \vec{c} = -\vec{a}$$

$$\Rightarrow \quad (\vec{b} + \vec{c})\cdot(\vec{b} + \vec{c}) = (-\vec{a})\cdot(-\vec{a})$$

$$\Rightarrow \quad \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{a}$$

$$\Rightarrow \quad |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$$

$$\Rightarrow \quad b^2 + c^2 + 2bc\cos(\pi - A) = a^2 \qquad \left(\because |\vec{x}|^2 = x^2\right)$$

$$\Rightarrow \quad b^2 + c^2 - 2bc\cos A = a^2$$

$$\Rightarrow \quad b^2 + c^2 - a^2 = 2bc\cos A$$

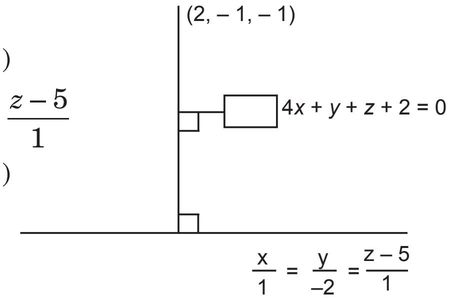
$$\Rightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

21. Equation of any line passing through the point (2, –1, –1) is

$$\frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c} \qquad ...(i)$$

Direction numbers of a normal to the plane $4x + y + z + 2 = 0$ are <4, 1, 1>

Since the line (*i*) is parallel to the given plane, therefore, the line (*i*) is perpendicular to a normal to the given plane.



$\therefore$ $4a + b + c = 0$ ...(*ii*)

Also the line (*i*) is perpendicular to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z-5}{1}$

$\therefore$ $a - 2b + c = 0$ ...(*iii*)

Now, $4a + b + c = 0$

$a - 2b + c = 0$

Solving (*ii*) and (*iii*) by cross-multiplication, we get

$$\frac{a}{1+2} = \frac{b}{1-4} = \frac{c}{-8-1} \quad \Rightarrow \quad \frac{a}{3} = \frac{b}{-3} = \frac{c}{-9}$$

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{-1} = \frac{c}{-3} \quad \Rightarrow \quad a : b : c = 1 : -1 : -3$$

Substituting these values in (*i*), the equation of the required line is

$$\frac{x-2}{1} = \frac{y+1}{-1} = \frac{z+1}{-3}$$

22. When a pair of dice is thrown, number of elements in sample space = 36, and all outcomes are equally likely.

Let E be event of throwing as 9, then

$$E = \{(3, 6), (6, 3), (4, 5), (5, 4)\} \text{ So } n(E) = 4$$

$\therefore$ Probability of throwing 9 = $\frac{4}{36} = \frac{1}{9}$

Probability of not throwing 9 = $1 - \frac{1}{9} = \frac{8}{9}$

As A starts the game, the probability of winning in the first throw = $\frac{1}{9}$ and so on.

A gets the second chance only if A and B both fail in the first round.

$\therefore$ Probability of A's winning in the 2nd round (i.e. third throw) = $\frac{8}{9} \cdot \frac{8}{9} \cdot \frac{1}{9} = \left(\frac{8}{9}\right)^2 \cdot \frac{1}{9}$

Similarly, probability of A's winning in the 3rd round (i.e. 5th throw) = $\left(\frac{8}{9}\right)^4 \cdot \frac{1}{9}$ and so on.

$\therefore$ The probability of A's winning the game is the sum of infinite series

$$\frac{1}{9} + \left(\frac{8}{9}\right)^2 \cdot \frac{1}{9} + \left(\frac{8}{9}\right)^4 \cdot \frac{1}{9} + \ldots\ldots$$

$$= \frac{1}{9}\left[1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^4 + \ldots\ldots \text{ to infinite terms}\right]$$, which is an infinite

Geometric series = $\frac{1}{9} \cdot \frac{1}{1-\left(\frac{8}{9}\right)^2} = \frac{1}{9} \cdot \frac{81}{81-64} = \frac{9}{17}$

$\therefore$ Probability of A getting the prize = $\frac{9}{17}$

23. Let H_1, H_2 and E be the events defined as follows:

H_1 = lost card is of clubs

H_2 = lost card is not of clubs and

E = two cards drawn are both of clubs.

Then $\quad P(H_1) = \frac{13}{52} = \frac{1}{4}$ and $P(H_2) = \frac{39}{52} = \frac{3}{4}$

When one card is lost, number of remaining cards in the pack = 51

When H_1 has occurred i.e. a card of clubs is lost, then the probability of drawing 2 cards of clubs from the remaining pack = $\frac{^{12}C_2}{^{51}C_2}$

So $$P\left(\frac{E}{H_1}\right) = \frac{^{12}C_2}{^{51}C_2} = \frac{12.11}{1.2} \times \frac{1.2}{51.50} = \frac{22}{425}$$

When H_2 has occurred i.e. when a card is not lost, then the probability of drawing 2 cards of clubs from the remaining pack = $\frac{^{13}C_2}{^{51}C_2}$

So $$P\left(\frac{E}{H_2}\right) = \frac{^{13}C_2}{^{51}C_2} = \frac{13.12}{1.2} \times \frac{1.2}{51.50} = \frac{26}{425}$$

We want to find $P\left(\frac{H_1}{E}\right)$

By Bayes' theorem, we have

$$P\left(\frac{H_1}{E}\right) = \frac{P(H_1) \cdot P\left(\frac{E}{H_1}\right)}{P(H_1)\, P\left(\frac{E}{H_1}\right) + P(H_2)\, P\left(\frac{E}{H_2}\right)} = \frac{\frac{1}{4} \cdot \frac{22}{425}}{\frac{1}{4} \cdot \frac{22}{425} + \frac{3}{4} \cdot \frac{26}{425}}$$

$$= \frac{22}{22+78} = \frac{22}{100} = \frac{11}{50}$$

24. Given $R = \{(a, b): a, b \in R$ and $a - b + \sqrt{3} \in S$, where S is the set of all irrational numbers$\}$

R is reflexive.

For every $a \in R$, $a - a + \sqrt{3} = \sqrt{3} \in S \Rightarrow (a, a) \in R$

$\Rightarrow R$ is reflexive.

R is not symmetric.

Take $a = \sqrt{3}$ and $b = 1$

As $\sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3} - 1 \in S$, $(\sqrt{3}, 1) \in R$

But $1 - \sqrt{3} + \sqrt{3} = 1 \notin S \Rightarrow (1, \sqrt{3}) \notin R$

$\Rightarrow R$ is not symmetric.

R is not transitive.

Take $a = 1$, $b = \sqrt{2}$ and $c = \sqrt{3}$

As $1 - \sqrt{2} + \sqrt{3} \in S$, $(1, \sqrt{2}) \in R$

Also $\sqrt{2} - \sqrt{3} + \sqrt{3} = \sqrt{2} \in S$, $(\sqrt{2}, \sqrt{3}) \in R$

But $1 - \sqrt{3} + \sqrt{3} = 1 \notin S$, $(1, \sqrt{3}) \notin R$

Thus, $(1, \sqrt{2}) \in R$ and $(\sqrt{2}, \sqrt{3}) \in R$ but $(1, \sqrt{3}) \notin R$

$\Rightarrow R$ is not transitive.

OR

The function f is one-one

Consider any $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + \alpha x_1 = 1 + \alpha x_2 \Rightarrow \alpha x_1 = \alpha x_2 \Rightarrow x_1 = x_2$ $(\because \alpha \neq 0)$

$\Rightarrow f$ is one-one

The function f is onto.

Consider any $y \in R$ (codomain of f)

Then $f(x) = y \Rightarrow 1 + \alpha x = y \Rightarrow x = \frac{y-1}{\alpha}$

As $y \in R$ and $\alpha \neq 0 \in R$, $\frac{y-1}{\alpha} \in R$

Thus, for all $y \in R$ (codomain of f), there exists $x = \frac{y-1}{\alpha} \in R$ (domain of f) such that

$$f(x) = f\left(\frac{y-1}{\alpha}\right) = 1+\alpha\left(\frac{y-1}{\alpha}\right) = 1 + y - 1 = y$$

Hence, the inverse function $f^{-1}: R \to R$ is given by $f^{-1}(x) = \frac{x-1}{\alpha}$

Further, we are given that $f^{-1} = f$

$\Rightarrow \quad f^{-1}(x) = f(x)$ for all $x \in R$

$\Rightarrow \quad \frac{x-1}{\alpha} = 1 + \alpha x$ for all $x \in R$

$\Rightarrow \quad \alpha + \alpha^2 x = x - 1$ for all $x \in R$

$\Rightarrow \quad \alpha^2 = 1$ and $\alpha = -1 \Rightarrow \alpha = -1$

25. Since, we know that the area of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad ...(i)$$

We know that, area of an equilateral triangle with side a

$$\Delta = \frac{\sqrt{3}}{4}a^2 \qquad ...(ii)$$

From (i) and (ii)

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4}a^2 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{2}a^2$$

$$\Rightarrow \quad 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{3}a^2$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix} = \sqrt{3}a^2$$

Squaring both sides

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$$

OR

Given $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0 \qquad \text{(Operate } R_2 \to R_2 - R_1\text{)}$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} = 0 \qquad \text{(Operate } R_3 \to R_3 - R_2\text{)}$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \sin A - \sin C & \sin B - \sin C & \sin C \\ \sin^2 A - \sin^2 C & \sin^2 B - \sin^2 C & \sin^2 C \end{vmatrix} = 0 \quad \text{(Operate } C_1 \to C_1 - C_3, C_2 \to C_2 - C_3\text{)}$$

[Take $(\sin A - \sin C)$ common from C_1 and $(\sin B - \sin C)$ common from C_2]

$$\Rightarrow (\sin A - \sin C)(\sin B - \sin C) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \sin C \\ \sin A + \sin C & \sin B + \sin C & \sin^2 C \end{vmatrix} = 0$$

$\Rightarrow (\sin A - \sin C)(\sin B - \sin C)(\sin B - \sin A) = 0$ (Expand by R_1)

$\Rightarrow \sin A = \sin C$ or $\sin B = \sin C$ or $\sin B = \sin A$

$\Rightarrow A = C$ or $B = C$ or $A = B$

$\Rightarrow \Delta ABC$ is an isosceles triangle.

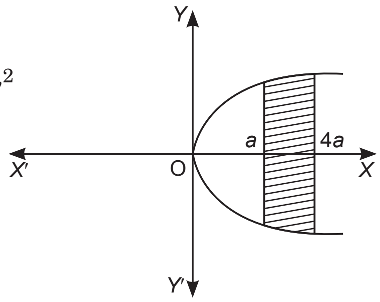
26. Given that $x = at^2$ ...(*i*)

and $y = 2at$ ...(*ii*)

Put $t = \dfrac{y}{2a}$ in (*i*) we get

$$x = a\left(\frac{y}{2a}\right)^2 \Rightarrow x = \frac{ay^2}{4a^2} \Rightarrow 4ax = y^2$$

$\Rightarrow \quad y^2 = 4ax$



Putting $t = 1$ and $t = 2$ in (*i*), we get

$$x = a \text{ and } x = 4a$$

Required area $= 2\int_a^{4a} y$ of the curve dx

$$= 2\int_a^{4a} \sqrt{4ax}\, dx = 2.2\int_a^{4a} \sqrt{ax}\, dx$$

$$= 4\sqrt{a}\int_a^{4a} \sqrt{x}\, dx = 4\sqrt{a}\int_a^{4a} x^{1/2}\, dx$$

$$= 4\sqrt{a} \cdot \frac{2}{3}\left(x^{3/2}\right)_a^{4a} = \frac{8\sqrt{a}}{3}[(4a)^{3/2} - a^{3/2}]$$

$$= \frac{8\sqrt{a}}{3}[4a\sqrt{4a} - a\sqrt{a}] \qquad (\because\ x^{3/2} = x\sqrt{x})$$

$$= \frac{8\sqrt{a}}{3}[(4a)\,2\sqrt{a} - a\sqrt{a}\,]$$

$$= \frac{8\sqrt{a}}{3}[8a\sqrt{a} - a\sqrt{a}\,] = \frac{8\sqrt{a}}{3} \cdot a\sqrt{a}\,[8-1]$$

$$= \frac{8}{3}a^2(7) = \frac{56}{3}a^2 \text{ sq. units}$$

27. Let $I = \int_3^5 \left(\sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}\right) dx$

Put $\sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}} = t$

Squaring both sides

$\Rightarrow \left(\sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}\right)^2 = t^2$

$\Rightarrow \left(\sqrt{x + 2\sqrt{2x-4}}\right)^2 + \left(\sqrt{x - 2\sqrt{2x-4}}\right)^2 + 2\sqrt{(x + 2\sqrt{2x-4})(x - 2\sqrt{2x-4})} = t^2$

$\Rightarrow x + 2\sqrt{2x-4} + x - 2\sqrt{2x-4} + 2\sqrt{x^2 - 4(2x-4)} = t^2$

$\Rightarrow 2x + 2\sqrt{x^2 - 8x + 16} = t^2$

$\Rightarrow 2x + 2\sqrt{(x-4)^2} = t^2$

$\Rightarrow 2x + 2(x-4) = t^2$

$\Rightarrow 2x + 2x - 8 = t^2$

$\Rightarrow 4x - 8 = t^2$

$\Rightarrow 4dx = 2t\,dt$

$\Rightarrow dx = \frac{1}{2}t\,dt$

When $x = 3$, $t = \sqrt{3 + 2\sqrt{2}} + \sqrt{3 - 2\sqrt{2}} = \sqrt{(\sqrt{2}+1)^2} + \sqrt{(\sqrt{2}-1)^2}$

$= \sqrt{2} + 1 + \sqrt{2} - 1 = 2\sqrt{2}$

When $x = 5$, $t = \sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}} = \sqrt{(\sqrt{3}+\sqrt{2})^2} + \sqrt{(\sqrt{3}-\sqrt{2})^2}$

$= \sqrt{3} + \sqrt{2} + \sqrt{3} - \sqrt{2} = 2\sqrt{3}$

$\therefore$ $I = \int_{2\sqrt{2}}^{2\sqrt{3}} t \cdot \frac{1}{2}t\,dt = \frac{1}{2}\int_{2\sqrt{2}}^{2\sqrt{3}} t^2\,dt = \frac{1}{6}\left[t^3\right]_{2\sqrt{2}}^{2\sqrt{3}}$

$I = \frac{1}{6}\left[(2\sqrt{3})^3 - (2\sqrt{2})^3\right] = \frac{4}{3}(3\sqrt{3} - 2\sqrt{2})$

OR

Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x}dx = \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}dx$ ($\because \cos 2A = \cos^2 A - \sin^2 A$)

$$I = \int \frac{\dfrac{\sqrt{\cos^2 x - \sin^2 x}}{\cos x}}{\dfrac{\sin x}{\cos x}}dx$$

$$I = \int \frac{\sqrt{\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}}}{\tan x} dx = \int \frac{\sqrt{1 - \tan^2 x}}{\tan x} dx$$

Put $1 - \tan^2 x = t^2$, $-2 \tan x \sec^2 x \, dx = 2t \, dt$, $dx = \dfrac{-t \, dt}{\tan x \sec^2 x}$ and $\tan^2 x = 1 - t^2$

$\therefore$ $$I = \int \frac{-t^2 \, dt}{\tan x \tan x \sec^2 x}$$

$$= \int \frac{-t^2 \, dt}{\tan^2 x (1 + \tan^2 x)} = \int \frac{-t^2}{(1 - t^2)(2 - t^2)} dt$$

Put $t^2 = y$

$$\frac{-t^2}{(1 - t^2)(2 - t^2)} = \frac{-y}{(1 - y)(2 - y)} = \frac{-1}{1 - y} + \frac{2}{2 - y} \qquad \text{(by partial fractions)}$$

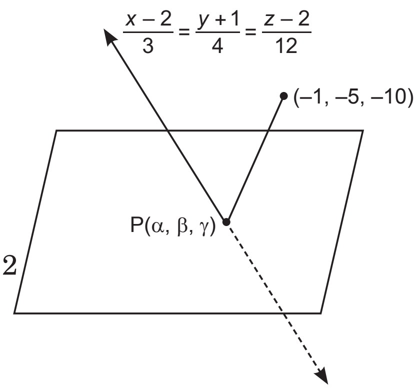
$$= \frac{-1}{1 - t^2} + \frac{2}{2 - t^2}$$

$\therefore$ $$I = \int \frac{-1 \, dt}{1 - t^2} + \int \frac{2}{(\sqrt{2})^2 - t^2} dt$$

$$= -\frac{1}{2} \log \left| \frac{1 + t}{1 - t} \right| + 2 \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t}{\sqrt{2} - t} \right| + C$$

$$= -\frac{1}{2} \log \left| \frac{1 + \sqrt{1 - \tan^2 x}}{1 - \sqrt{1 - \tan^2 x}} \right| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + C$$

28. Let $P(\alpha, \beta, \gamma)$ be the point of intersection of the given line (*i*) and plane (*ii*)



$$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{12} \qquad \ldots(i)$$

And $x - y + z = 5$ ...(*ii*)

$\Rightarrow$ $\dfrac{\alpha - 2}{3} = \dfrac{\beta + 1}{4} = \dfrac{\gamma - 2}{12} = \lambda$ (say)

$\Rightarrow$ $\alpha = 3\lambda + 2, \beta = 4\lambda - 1, \gamma = 12\lambda + 2$

Also, point $P(\alpha, \beta, \gamma)$ lie on plane (*ii*)

$\Rightarrow$ $\alpha - \beta + \gamma = 5$ ...(*iii*)

Putting the value of α, β, γ in (*iii*) we get

$\Rightarrow$ $3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 5$

$\Rightarrow$ $11\lambda + 5 = 5$

$\Rightarrow$ $11\lambda = 0$

$\Rightarrow$ $\lambda = 0$

$\Rightarrow$ $\alpha = 2; \beta = -1; \gamma = 2$

Hence, the coordinate of the point of intersection P is $(2, -1, 2)$

Therefore, required distance $= d = \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$ units

29. Let x kg and y kg be of two types of fertilisers A and B, respectively.

	x	y	At least
Nitrogen	12%	4%	12 kg
Phosphoric	5%	5%	12 kg

Minimum cost C = $10x + 8y$

Subject to the constraint

$$\frac{12}{100}x + \frac{4}{100}y \geq 12 \text{ or } 3x + y \geq 300$$

$$\frac{5}{100}x + \frac{5}{100}y \geq 12 \text{ or } 5x + 5y \geq 1200$$

and $\quad x, y \geq 0$

Now, $\quad 3x + y = 300 \quad \ldots(i)$

$$\Rightarrow \quad \frac{3x}{300} + \frac{y}{300} = 1$$

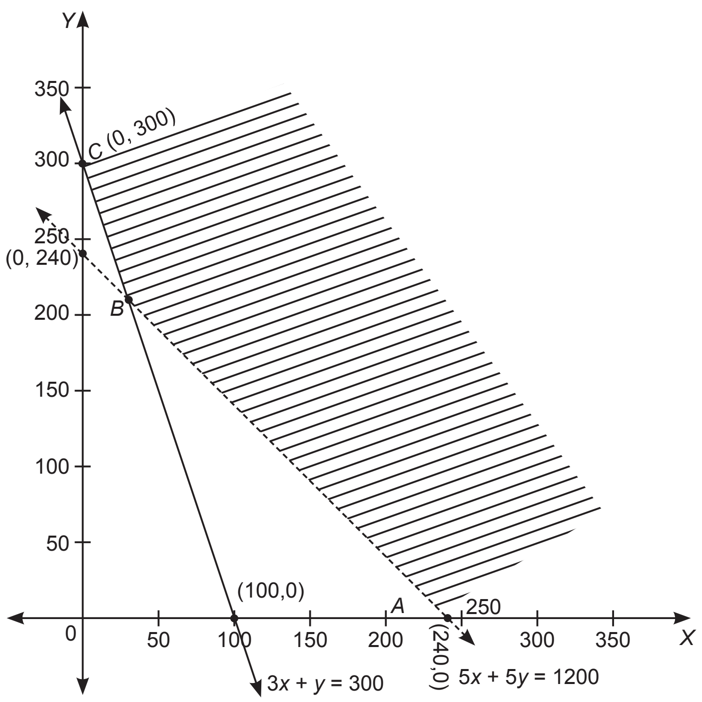
$$\Rightarrow \quad \frac{x}{100} + \frac{y}{300} = 1$$

Point on x-axis (100, 0) and y-axis (0, 300)

and $\quad 5x + 5y = 1200$

$$\Rightarrow \quad \frac{5x}{1200} + \frac{5y}{1200} = 1 \quad \Rightarrow \quad \frac{x}{240} + \frac{y}{240} = 1$$

Point on x-axis (240, 0) and y-axis (0, 240)



Corner point method: The coordinates of the corner point A, B and C of the feasible region ABC are (240, 0), (30, 210) and (0, 300) respectively.

Note that the coordinates of B are obtained by solving $3x + y = 300$ and $5x + 5y = 1200$

The value of the objective function at these are given in the following table:

Corner points (*x*, *y*) of the region *ABC*	**$C = 10x + 8y$**
$A(240, 0)$	$10(240) + 8(0) = 2400$
$B(30, 210)$	$10(30) + 8(210) = 1980$
$C(0, 300)$	$10(0) + 8(300) = 2400$

We see that the point $B(30, 210)$ is giving minimum value of cost.

Hence, 30 kg of type A and 210 kg of type B fertilisers are used so that nutrient requirements are met at minimum cost ₹ 1980.