Solutions to RSPL/1

1. $$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + 2C_1$, we get

$$\begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

2. $$f(x) = x\,|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-h^2 - 0}{-h} = \lim_{h \to 0} h = 0$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 - 0}{h} = \lim_{h \to 0} h = 0$$

$$Lf'(0) = Rf'(0).$$

Hence, differentiable at $x = 0$.

3. $$\int_1^3 \frac{\cos(\log x)}{x}\,dx = \int_1^{\log 3} \cos t\,dt = \Big[\sin t\Big]_0^{\log 3}$$

$$= \sin(\log 3) - \sin(0)$$

$$= \sin(\log 3)$$

Let $\log x = t \Rightarrow \frac{1}{x}dx = dt$
When $x = 1$, $t = 0$
and when $x = 3$, $t = \log 3$

4. Given planes are $2x + y + 2z - 8 = 0$, i.e. $2x + y + 2z - 8 = 0$

and $4x + 2y + 4z + 5 = 0$, i.e. $2x + y + 2z + \frac{5}{2} = 0$

$$\therefore \quad \text{Distance} = \left|\frac{\frac{5}{2} + 8}{\sqrt{4+1+4}}\right| = \left|\frac{5+16}{2 \times 3}\right| = \frac{21}{6} = \frac{7}{2} \text{ units.}$$

5. $$2X + Y = 0 \quad \Rightarrow \quad 2X = 0 - Y$$

$$\Rightarrow \quad X = -\frac{1}{2}Y \quad \Rightarrow \quad X = -\frac{1}{2}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} -\frac{3}{2} & -1 \\ -\frac{1}{2} & -2 \end{bmatrix}$$

6. Consider $y = \cos(\sin x^2)$

$$\frac{dy}{dx} = -\sin(\sin x^2) \cdot \frac{d}{dx}(\sin x^2) = -\sin(\sin x^2) \cdot \cos x^2 \cdot \frac{d}{dx}(x^2)$$

$$= -\sin(\sin x^2) \cdot \cos x^2 \cdot 2x$$

$$= -2x \cdot \sin(\sin x^2) \cdot \cos x^2$$

$$\left.\frac{dy}{dx}\right]_{x = \sqrt{\frac{\pi}{2}}} = -2 \cdot \sqrt{\frac{\pi}{2}} \cdot \sin\left(\sin \frac{\pi}{2}\right) \cos \frac{\pi}{2} = -\sqrt{2\pi} \cdot \sin(1) \cdot 0 = 0$$

7. Let side of an equilateral triangle be x cm.

$\therefore$ $$\frac{dx}{dt} = 2 \text{ cm/s and } \frac{dA}{dt} = 20\sqrt{3} \text{ cm}^2\text{/s}$$

Area $$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4}\cdot 2x \cdot \frac{dx}{dt} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x \times 2$$

$$\Rightarrow \quad 20\sqrt{3} = \sqrt{3}x \Rightarrow x = \frac{20\sqrt{3}}{\sqrt{3}} = 20$$

Hence, side is 20 cm.

8. Given $f(x) = (x-1)^{2/3}$ in [0, 2]

Continuous in [0, 2] as algebraic expression with positive exponent is continuous.

$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}$$

$f'(1)$ does not exist at $x = 1$ not differentiable at $x = 1 \in [0, 2]$

Hence, Rolle's Theorem is not applicable.

9. Given line is $$\frac{x+2}{2} = \frac{2y-7}{6} = \frac{5-z}{6}$$

$$\Rightarrow \quad \frac{x+2}{2} = \frac{y-\frac{7}{2}}{3} = \frac{z-5}{-6}$$

$\therefore$ DR's are 2, 3, –6

Dividing by $\sqrt{4+9+36} = 7$

$\therefore$ Direction cosines of the line are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$.

10. $$P(A-B) = P(A) - P(A \cap B)$$

$$\Rightarrow \quad \frac{3}{8} = \frac{3}{4} - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

Now, $$P(A)\,.\,P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} = P(A \cap B)$$

$\therefore$ A and B are independent events.

11. Let x large vans and y small vans be used.

LPP is to minimise cost $Z = 400x + 200y$

Subject to constraints

$$x \ge 0, y \ge 0$$

$$200x + 80y \le 1200$$

$$x \le y$$

$$400x + 200y \le 3000$$

12. Consider $\int \frac{x^4+3}{x^2+1}dx = \int \frac{(x^4-1)+4}{x^2+1}dx = \int \frac{(x^2-1)(x^2+1)+4}{x^2+1}dx$

$$= \int \left(x^2 - 1 + \frac{4}{x^2+1}\right)dx$$

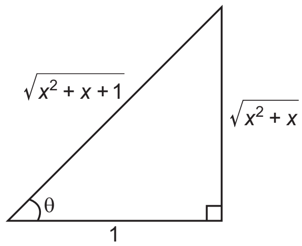
$$= \frac{x^3}{3} - x + 4\tan^{-1}x + C$$

13. Consider $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ ...(*i*)

$\Rightarrow \quad \tan^{-1}\sqrt{x(x+1)} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1} = \cos^{-1}\sqrt{x^2+x+1}$

Let $\tan^{-1}\sqrt{x(x+1)} = \theta \Rightarrow \tan\theta = \sqrt{x^2+x}$

$\therefore \quad \cos\theta = \frac{1}{\sqrt{x^2+x+1}}$



$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right)$ ...(*ii*)

From (*i*) and (*ii*), we get

$$\cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) = \cos^{-1}\sqrt{x^2+x+1}$$

$\Rightarrow \quad \frac{1}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1}$

$\Rightarrow \quad x^2 + x + 1 = 1$

$\Rightarrow \quad x(x+1) = 0 \Rightarrow x = 0, -1$

Both the values satisfy (*i*). Hence, $x = 0, -1$.

14. Consider $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$ [By performing $C_1 \to C_1 + (C_2 + C_3)$]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$ [By taking $(a+b+c)$ common from C_1]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$ [By performing $R_3 \to R_3 - 2R_1$]

$= (a+b+c)[(b-c)(a+b-2c) - (c-a)(c+a-2b)]$

$= (a+b+c)(ab + b^2 - 2bc - ac - bc + 2c^2 - c^2 + a^2 + 2bc - 2ab)$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$

15. Let $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ | Let $x = \sin\theta$ $\Rightarrow \theta = \sin^{-1}x$

$y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$= \tan^{-1}(\tan\theta) = \theta$

$\Rightarrow \quad y = \sin^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ ...(*i*)

and $t = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ | Let $x = \cos\theta$ $\Rightarrow \theta = \cos^{-1} x$

$t = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= \sec^{-1}(\sec 2\theta) = 2\theta$

$\Rightarrow \quad t = 2\cos^{-1} x$

$\frac{dt}{dx} = -\frac{2}{\sqrt{1-x^2}}$ ...(*ii*)

$\therefore \quad \frac{dy}{dt} = \frac{dy}{dx} \div \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{(-2)} = -\frac{1}{2}$ [From (*i*) and (*ii*)]

OR

Consider $y = x^3 \log\left(\frac{1}{x}\right) = -x^3 \log x$ ...(*i*)

On differentiating both sides w.r.t. *x*, we get

$$\frac{dy}{dx} = -x^3 \cdot \frac{1}{x} - 3x^2 \log x$$

$$\Rightarrow \quad x\frac{dy}{dx} = -x^3 - 3x^3 \log x$$

$$\Rightarrow \quad x\frac{dy}{dx} = -x^3 + 3y$$ [From (*i*)]

Again differentiating w.r.t. *x*, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -3x^2 + 3\frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 3x^2$$

Dividing throughout by *x*, we get

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + 3x = 0$$

16. Consider $\int \frac{dx}{5 + 4\cos x}dx = \int \frac{1}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}dt$

$$= \int \frac{(1+t^2)}{5 + 5t^2 + 4 - 4t^2} \times \frac{2}{1+t^2}dt$$

$$= 2\int \frac{1}{9+t^2}dt$$

$$= \frac{2}{3}\tan^{-1}\frac{t}{3} + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C$$

Let $\tan\frac{x}{2} = t$

$\Rightarrow x = 2\tan^{-1} t$

$\Rightarrow dx = \frac{2}{1+t^2}dt$

$\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$

17. Consider $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cdot\sec^2\frac{x}{2}\,dx = \frac{1}{2}\left[x\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}} - \int 1\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right]_0^{\pi/2}$

$$= \left[x\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}\,dx = \left[x\tan\frac{x}{2} - 2\log\left|\sec\frac{x}{2}\right|\right]_0^{\frac{\pi}{2}}$$

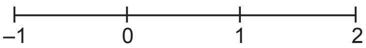
$$= \left(\frac{\pi}{2}\tan\frac{\pi}{4} - 2\log\left|\sec\frac{\pi}{4}\right|\right) - (0 - 2\log 1)$$

$$= \frac{\pi}{2} - 2\log\sqrt{2} = \frac{\pi}{2} - \log 2$$

OR

$$x^3 - x = x(x-1)(x+1)$$

For $-1 < x < 0$, $x^3 - x = (-)(-)(+) > 0$



For $0 < x < 1$, $x^3 - x = (+)(-)(+) < 0$

For $1 < x < 2$, $x^3 - x = (+)(+)(+) > 0$

$\therefore$ $\int_{-1}^{2}|x^3 - x|\,dx = \int_{-1}^{0}(x^3 - x)\,dx + \int_0^1 -(x^3 - x)\,dx + \int_1^2 (x^3 - x)\,dx$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_1^2$$

$$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) - 0 + \left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$$

18. $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0 \Rightarrow \sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$

$\Rightarrow$ $\sqrt{1+x^2}\sqrt{1+y^2} = -xy\frac{dy}{dx} \Rightarrow \int\frac{y}{\sqrt{1+y^2}}dy = -\int\frac{\sqrt{1+x^2}}{x}dx$ ... (i)

For $\int\frac{y}{\sqrt{1+y^2}}dy = \frac{1}{2}\int\frac{1}{\sqrt{t}}dt = \frac{1}{2}\times 2\sqrt{t} + C_1 = \sqrt{1+y^2} + C_1$ ...(ii) | Let $1 + y^2 = t$ $\Rightarrow 2y\,dy = dt$

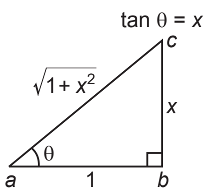
For $\int\frac{\sqrt{1+x^2}}{x}dx$, | Let $\tan\theta = x$ $\Rightarrow \sec^2\theta\,d\theta = dx$

$\int\frac{\sec\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta = \int\frac{1}{\sin\theta\cdot\cos^2\theta}d\theta = \int\frac{\sin\theta}{\sin^2\theta\cos^2\theta}d\theta$ | Let $\cos\theta = t$ $\Rightarrow -\sin\theta\,d\theta = dt$

$$= -\int\frac{1}{t^2(1-t^2)}dt = -\int\left(\frac{1}{1-t^2} + \frac{1}{t^2}\right)dt = -\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{t} + C_2$$

$$= -\frac{1}{2}\log\left|\frac{1+\cos\theta}{1-\cos\theta}\right| + \frac{1}{\cos\theta} + C_2$$

$$= -\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right| + \sqrt{1+x^2} + C_2 \quad ...(iii)$$



Substituting from (*ii*) and (*iii*) in (*i*), we get

$$\sqrt{1+y^2} = \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}\right| - \sqrt{1+x^2} + C \text{ is the required solution.}$$

where C (constant) = $C_1 + C_2$

19. Let $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$ ...(*i*)

$$\vec{a}\cdot\vec{b} = 14 \Rightarrow (2\hat{i}+2\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = 14$$

$\Rightarrow$ $2x + 2y + z = 14$ ...(*ii*)

Also $\vec{a}\times\vec{b} = 3\hat{i} + \hat{j} - 8\hat{k}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ x & y & z \end{vmatrix} = 3\hat{i} + \hat{j} - 8\hat{k}$$

$$\Rightarrow (2z-y)\hat{i} - (2z-x)\hat{j} + (2y-2x)\hat{k} = 3\hat{i} + \hat{j} - 8\hat{k}$$

$\Rightarrow$ $2z - y = 3$ ...(*iii*)

$x - 2z = 1$ ...(*iv*)

and $2y - 2x = -8$ ...(*v*)

From (*ii*) and (*v*), $4y + z = 6$ ...(*vi*)

From (*iii*) and (*vi*), we get

$$4(2z-3) + z = 6 \Rightarrow 8z - 12 + z = 6 \Rightarrow 9z = 18 \Rightarrow z = 2$$

From (*iv*) and (*vi*), $y = 1,\ x = 5$

$\therefore$ $\vec{b} = 5\hat{i} + \hat{j} + 2\hat{k}$

OR

$$|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = (|\vec{a}||\hat{i}|\sin\alpha)^2 + (|\vec{a}||\hat{j}|\sin\beta)^2 + (|\vec{a}||\hat{k}|\sin\gamma)^2$$

$$= |\vec{a}|^2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$$

$$= a^2(1-\cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma)$$

$$= a^2(3 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma)$$

$$= a^2(3-1) = 2a^2$$

20. Consider relations

$$3l + m + 5n = 0$$

$\Rightarrow$ $m = -3l - 5n$ ...(*i*)

and $6mn - 2nl + 5lm = 0$ ...(*ii*)

$\Rightarrow$ $6n(-3l-5n) - 2nl + 5l(-3l-5n) = 0$

$\Rightarrow$ $-18nl - 30n^2 - 2nl - 15l^2 - 25ln = 0$

$\Rightarrow$ $15l^2 + 45ln + 30n^2 = 0 \Rightarrow l^2 + 3ln + 2n^2 = 0$

$$\Rightarrow \left(\frac{l}{n}\right)^2 + 3\left(\frac{l}{n}\right) + 2 = 0 \Rightarrow \left(\frac{l}{n}+1\right)\left(\frac{l}{n}+2\right) = 0$$

$\Rightarrow \quad \frac{l}{n} + 1 = 0$ or $\frac{l}{n} + 2 = 0$

$\Rightarrow \quad l = -n$ or $l = -2n$

From (i), $m = 3n - 5n = -2n$ | From (i), $m = 6n - 5n = n$

$\frac{l}{-1} = \frac{m}{-2} = \frac{n}{1}$ | $\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$

DR's : –1, –2, 1 | DR's : –2, 1, 1

$\therefore$ DR's are 1, 2, –1 or –2, 1, 1

$$\therefore \quad \cos\theta = \frac{-2+2-1}{\sqrt{1+4+1}\,\sqrt{4+1+1}} = \frac{-1}{6}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{6}\right)$$

21. **Case I:** 2 white balls and 2 other coloured balls in bag

Case II: 3 white balls and 1 other coloured ball in bag

Case III: 4 white balls and no other coloured ball in bag

$$P(I) = P(II) = P(III) = \frac{1}{3}$$

E: 2 white balls are drawn

$$P(E/I) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6};\ P(E/II) = \frac{{}^3C_2}{{}^4C_2} = \frac{3}{6};\ P(E/III) = \frac{{}^4C_2}{{}^4C_2} = 1$$

Using Bayes' Theorem, probability of drawing 2 white balls from bag when all the balls are white

$$P(III/E) = \frac{P(III).\,P(E/III)}{P(I).P(E/I) + P(II).P(E/II) + P(III).P(E/III)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times\frac{1}{6} + \frac{1}{3}\times\frac{3}{6} + \frac{1}{3}\times 1}$$

$$= \frac{1}{\frac{1}{6} + \frac{3}{6} + 1} = \frac{6}{10} = \frac{3}{5}$$

22. A: person has TB $\quad$ B: person does not have TB

(1 in 1000 persons who have TB)

$$\therefore \quad P(A) = \frac{1}{1000},\ P(B) = \frac{999}{1000}$$

E: person is diagnosed to have TB

$$P(E/A) = 0.99;\ P(E/B) = 0.001$$

Using Bayes' Theorem, probability that person actually has TB when he is diagnosed to have TB.

$$P(A/E) = \frac{P(A)\cdot P(E/A)}{P(A)\cdot P(E/A)+P(B)\cdot P(E/B)}$$

$$= \frac{\frac{1}{1000}\times 0.99}{\frac{1}{1000}\times 0.99+\frac{999}{1000}\times 0.001}$$

$$= \frac{990}{990+999} = \frac{990}{1989} = \frac{110}{221}$$

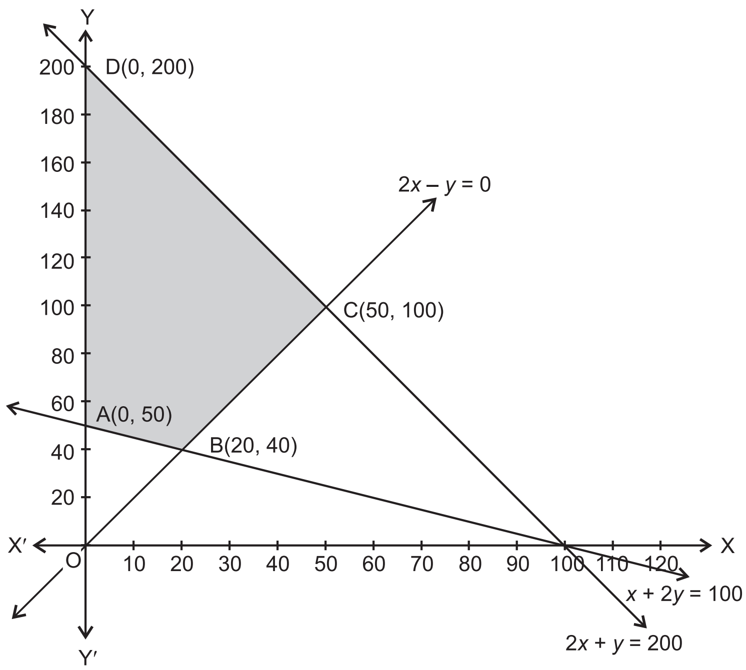
Wear mask or use cloth when coughing. Medicines must be taken as directed by doctor.

23. Draw graph of inequalities $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x \geq 0$, $y \geq 0$.

The feasible region determined by the constraints, $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x \geq 0$ and $y \geq 0$ is shown;

ABCDA is the feasible region.

The corner points of the feasible region are $A(0, 50)$, $B(20, 40)$, $C(50, 100)$ and $D(0, 200)$.



The values of Z at these corner points are as follows:

Corner points	$Z = x + 2y$	
$A(0, 50)$	100	← Minimum
$B(20, 40)$	100	← Minimum
$C(50, 100)$	250	
$D(0, 200)$	400	← Maximum

The maximum value of Z is 400 at (0, 200) and the minimum value of Z is 100 at all the points on the line segment joining the points (0, 50) and (20, 40).

24. Consider $$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$$

Matrix formed by cofactors of each element in $|A|$

$$\begin{bmatrix} A_{11} = \cos\alpha & A_{12} = -\sin\alpha & A_{13} = 0 \\ A_{21} = \sin\alpha & A_{22} = \cos\alpha & A_{23} = 0 \\ A_{31} = 0 & A_{32} = 0 & A_{33} = 1 \end{bmatrix}$$

$\therefore$ $$\text{Adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}' = \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|}\text{ Adj } A = \frac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider $$AA^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & 0-0+0 \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

OR

Given $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$; $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB$$

$$(A+B)^2 = A^2 + AB + BA + B^2$$

Given $(A+B)^2 = A^2 + B^2$

$\therefore$ $AB + BA = O$ ...(*i*)

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$$

From (*i*), $\begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} + \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $$\begin{bmatrix} 2a-b+2 & 1-a \\ 2a-2 & 4-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $2a - b + 2 = 0;\quad 1 - a = 0 \quad \Rightarrow \quad a = 1$

$2a - 2 = 0;\quad 4 - b = 0 \quad \Rightarrow \quad b = 4$

Hence, $a = 1$, $b = 4$ satisfies all four equations.

25. For reflexive: Let for $(a, b) \in A \times A$

$(a, b)\, R\, (a, b) \Rightarrow a + b = b + a$, true. Hence, reflexive.

For symmetric: Let for $(a, b), (c, d) \in A \times A$

$(a, b)\, R\, (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\, R\, (a, b)$

Hence, symmetric.

For transitive: Let for $(a, b), (c, d), (e, f) \in A \times A$

$\Rightarrow (a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$

$\Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow a + d + c + f = b + c + d + e \quad \Rightarrow \quad a + f = b + e$

$\Rightarrow (a, b)\, R\, (e, f)$. Hence, R is transitive

As relation R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

Equivalence class $[(3, 4)]$

$\Rightarrow \quad (a, b)\, R\, (3, 4) \quad \Rightarrow \quad a + 4 = b + 3$

Give $A = \{1, 2, 3, ..., 10\}$

$\therefore$ Ordered pairs can be (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10)

$\therefore$ $[(3, 4)] = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10)\}$

OR

Given $f : N \to Y$ where $Y = \{n^2 : n \in N\}$ defined by $f(n) = n^2$.

To show f is invertible, we have to show that it is bijective.

For one-one: Let for $n_1, n_2 \in N$

$f(n_1) = f(n_2) \Rightarrow n_1^2 = n_2^2 \Rightarrow n_1 = n_2$

Hence, one-one

For onto: Let for some $y \in Y$, there exists $n \in N$ such that

$y = n^2 \Rightarrow n = \sqrt{y} \in N$.

Hence, onto.

As function is one-one and onto. Hence, f is invertible.

We define a function $g : Y \to N$ as $g(n) = \sqrt{n}$.

Consider $fog(n) = f(g(n)) = f(\sqrt{n}) = (\sqrt{n})^2 = n$

and $gof(n) = g(f(n)) = g(n^2) = \sqrt{n^2} = n$

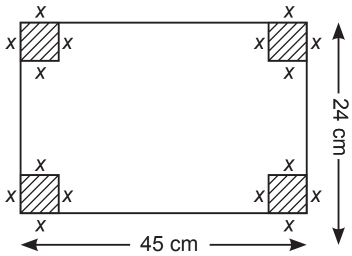
As $fog = I_Y$ and $gof = I_N$

$\Rightarrow$ f is invertible and g is inverse of f

Hence, $f^{-1}(n) = \sqrt{n}$

26. Let square of side x cm be cut off from all corners. Then

$$l = 45 - 2x,\ b = 24 - 2x,\ h = x$$



$\therefore$ Volume $(V) = (45 - 2x)(24 - 2x)x$

$$= (45 - 2x)(24x - 2x^2)$$

$$\frac{dV}{dx} = (45 - 2x)(24 - 4x) + (24x - 2x^2)(-2)$$

$$= 1080 - 180x - 48x + 8x^2 - 48x + 4x^2$$

$$= 12x^2 - 276x + 1080 = 12(x^2 - 23x + 90)$$

$$= 12(x - 18)(x - 5)$$

For maximum V, $\frac{dV}{dx} = 0 \Rightarrow x - 18 = 0$ or $x - 5 = 0 \Rightarrow x = 5, 18$, ($x = 18$ not possible)

$$\frac{d^2V}{dx^2} = 12[(x - 18)\cdot 1 + (x - 5)\cdot 1]$$

$$\left.\frac{d^2V}{dx^2}\right]_{x=5} = 12(-13 + 0) < 0$$

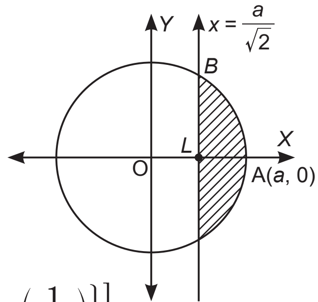
$\therefore$ For $x = 5$, volume is maximum.

Hence, a square of side 5 cm must be cut off to make a base of maximum volume.

27. Given the circle $x^2 + y^2 = a^2$ and the line $x = \frac{a}{\sqrt{2}}$.

On plotting the curve and line, we notice we have to find area of shaded portion.

As the curve is symmetrical to the x-axis.



$\therefore$ Area $= 2 \times$ area (LBA)

$$= 2\int_{\frac{a}{\sqrt{2}}}^{a} y\,dx = 2\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= 2\left[\left(\frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}1\right) - \left\{\frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right\}\right]$$

$$= 2\left(\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} - \frac{a^2}{2}\cdot\frac{\pi}{4}\right)$$

$$= 2\left(\frac{\pi a^2}{8} - \frac{a^2}{4}\right) = \left(\frac{\pi a^2}{4} - \frac{a^2}{2}\right) \text{ sq units.}$$

28. Consider equation $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$ Hence, homogeneous.

As $$F\left(\frac{\lambda y}{\lambda x}\right) = \frac{\lambda y}{\lambda x} - \tan\left(\frac{\lambda y}{\lambda x}\right) = \frac{y}{x} - \tan\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = v - \tan v \Rightarrow \int \frac{1}{\tan v} dv = -\int \frac{1}{x} dx$$

$$\Rightarrow \quad \int \cot v \, dv = -\int \frac{1}{x} dx \Rightarrow \log|\sin v| = -\log|x| + \log C$$

$$\Rightarrow \quad \log|\sin v| = \log\left|\frac{C}{x}\right| \Rightarrow x \sin\frac{y}{x} = C \text{ is the required solution.}$$

29. If lines are coplanar, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Here $x_1 = a - d, y_1 = a, z_1 = a + d;$

$a_1 = \alpha - \delta, b_1 = \alpha, c_1 = \alpha + \delta$

and $x_2 = b - c, y_2 = b, z_2 = b + c;$

$a_2 = \beta - \gamma, b_2 = \beta, c_2 = \beta + \gamma.$

$$\Rightarrow \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

By performing $C_1 \to C_1 + C_3$, we get

$$\begin{vmatrix} 2b-2a & b-a & b+c-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\Rightarrow 2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0,$$

True, as C_1 and C_2 are identical.

Hence, the lines are coplanar.

OR

Plane determined by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$ is

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$\Rightarrow (x - 2)(16) - (y - 5)(-24) + (z + 3)(32) = 0$

$$\Rightarrow \qquad 16x - 32 + 24y - 120 + 32z + 96 = 0$$

$$\Rightarrow \qquad 16x + 24y + 32z - 56 = 0$$

$$\Rightarrow \qquad 2x + 3y + 4z - 7 = 0$$

Distance of the point $P(7, 2, 4)$ from the plane $2x + 3y + 4z - 7 = 0$ is

$$\left|\frac{2 \times 7 + 3 \times 2 + 4 \times 4 - 7}{\sqrt{4 + 9 + 16}}\right| = \left|\frac{14 + 6 + 16 - 7}{\sqrt{29}}\right| = \sqrt{29} \text{ units.}$$