Solutions to RSPL/2

1. Let, $$I = \int_0^{\frac{\pi}{2}} \log(\tan x)\, dx \quad \text{...}(i)$$

Using $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \log\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} dx = \int_0^{\frac{\pi}{2}} \log(\cot x)\, dx \quad \text{...}(ii)$$

On adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \log(\tan x \cot x)\, dx = \int_0^{\frac{\pi}{2}} \log 1\, dx = 0 \Rightarrow I = 0$$

2. Given $$f(x) = |x+3| = \begin{cases} x+3, & x \geq -3 \\ -x-3, & x < -3 \end{cases}$$

For continuity at $x = -3$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = f(-3)$$

$$\lim_{x \to -3^-} (-x-3) = \lim_{x \to -3^+} (x+3) = 0$$

$$\Rightarrow \quad 3 - 3 = -3 + 3 = 0, \text{ true}$$

$\therefore$ Continuous at $x = -3$.

3.
$$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & -\sin\alpha\cos\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha + \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & \cos 2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

4. $|\vec{a}| = 1, |\vec{b}| = 1$

If $|\vec{a} - \sqrt{2}\,\vec{b}| = 1 \Rightarrow |\vec{a} - \sqrt{2}\vec{b}|^2 = 1 \Rightarrow (\vec{a} - \sqrt{2}\vec{b})^2 = 1$

$\Rightarrow \quad \vec{a}^2 + 2\vec{b}^2 - 2\sqrt{2}\,\vec{a}\cdot\vec{b} = 1 \Rightarrow |\vec{a}|^2 + 2|\vec{b}|^2 - 2\sqrt{2}\,\vec{a}\cdot\vec{b} = 1$

$\Rightarrow \quad 1 + 2 - 2\sqrt{2}\,\vec{a}\cdot\vec{b} = 1 \Rightarrow 2\sqrt{2}\,\vec{a}\cdot\vec{b} = 2 \Rightarrow \vec{a}\cdot\vec{b} = \dfrac{1}{\sqrt{2}}$

$$\therefore \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

5. Given area = 4 sq units

$$\therefore \quad \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 4$$

$\Rightarrow \quad -2(4-k) + 1(0) = \pm 4 \Rightarrow -8 + 2k = \pm 4$

$\Rightarrow \quad 2k = \pm 4 + 8 \Rightarrow 2k = 12, 4 \Rightarrow k = 6, 2$

6. Consider $$y = \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)$$

$$\frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} - x\right)\cdot(-1)$$

$$\Rightarrow \quad \frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$$

7. Let r be radius of a sphere

$$r = 9 \text{ cm and } \Delta r = 0.03 \text{ cm}$$

To find ΔS, where S is surface area of a sphere

$$\Delta S = \frac{dS}{dr}\cdot\Delta r = \frac{d}{dr}(4\pi r^2)\cdot\Delta r = 8\pi r\cdot\Delta r$$

$$\Delta S = 8\pi \times 9 \times 0.03 = 8\pi \times 0.27 = 2.16\pi \text{ cm}^2$$

8. Given $$f(x) = x^2 + ax + 1$$

$$f'(x) = 2x + a \quad \text{...}(i)$$

Given interval is (1, 2), i.e. $1 < x < 2$

$$1 < x < 2 \Rightarrow 2 < 2x < 4$$

add a, $$2 + a < 2x + a < 4 + a$$

$$\Rightarrow \quad 4 + a > f'(x) > 2 + a \quad \text{[From } (i)] \text{ ...}(ii)$$

For increasing $f'(x) > 0$

$\therefore$ For increasing at least $2 + a = 0 \Rightarrow a = -2$.

9. $[\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow$ One vector can be written as a linear combination of other two.

$\therefore$ $\vec{a} = \lambda\vec{b} + \mu\vec{c}$, for some scalars λ, μ.

10. $$P(A \cup B) = p, \quad P(A \cap B) = r$$

$$P(\text{exactly one occur}) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$
$$= P(A \cup B) - P(A \cap B) = p - r$$

11. Given equation $\frac{dy}{dx} + \cos\left(\frac{dy}{dx}\right) = 0$

Order is 1 and degree is not defined as equation cannot be written as a polynomial of differentials.

12. Consider $\int \frac{\sqrt{x}}{x+1}dx = \int \frac{t}{t^2+1}\cdot 2t\,dt$ | Let $x = t^2$ $\Rightarrow dx = 2t\,dt$

$$= 2\int \frac{(t^2+1)-1}{t^2+1}dt$$

$$= 2\left[\int 1\cdot dt - \int \frac{1}{t^2+1}dt\right]$$

$$= 2(t - \tan t) + C = 2[\sqrt{x} - \tan^{-1}\sqrt{x}] + C$$

13. Consider $\tan^{-1}\left(\frac{1}{1+1.2}\right)+\tan^{-1}\left(\frac{1}{1+2.3}\right)+\tan^{-1}\left(\frac{1}{1+3.4}\right)+\ldots+\tan^{-1}\left(\frac{1}{1+n(n+1)}\right)$

$= \tan^{-1}\left(\frac{2-1}{1+1.2}\right)+\tan^{-1}\left(\frac{3-2}{1+2.3}\right)+\tan^{-1}\left(\frac{4-3}{1+3.4}\right)+\ldots+\tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$

$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + (\tan^{-1} 4 - \tan^{-1} 3) + \ldots + [\tan^{-1}(n+1) - \tan^{-1} n]$

$= \tan^{-1} 2 - \tan^{-1} 1 + \tan^{-1} 3 - \tan^{-1} 2 + \tan^{-1} 4 - \tan^{-1} 3 + \ldots + \tan^{-1}(n+1) - \tan^{-1} n$

$= \tan^{-1}(n+1) - \tan^{-1} 1$

$= \tan^{-1}\left[\frac{(n+1)-1}{1+(n+1)\cdot 1}\right] = \tan^{-1}\left(\frac{n}{n+2}\right)$

Hence, $\tan^{-1}\left(\frac{n}{n+2}\right) = \tan^{-1} x$

$\Rightarrow x = \frac{n}{n+2}$.

14. Given $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$

Consider $A^2 + xI = yA$

$\Rightarrow \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix} \Rightarrow \begin{bmatrix} 16+x & 8 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$

$\Rightarrow 16 + x = 3y, \quad y = 8, \quad 56 = 7y, \quad 32 + x = 5y$

$\Rightarrow x = 8, \quad y = 8$

$\therefore$ We have $A^2 + 8I = 8A$

Post multiply by A^{-1}, we get

$$A^2A^{-1} + 8IA^{-1} = 8AA^{-1}$$

$$\Rightarrow \quad A(AA^{-1}) + 8A^{-1} = 8I \Rightarrow AI + 8A^{-1} = 8I$$

$$\Rightarrow \quad A + 8A^{-1} = 8I \Rightarrow A^{-1} = \frac{1}{8}(8I - A)$$

$$A^{-1} = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$$

OR

Consider $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} A \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$B\ A\ C = I$$

$$\Rightarrow \quad A = B^{-1}IC^{-1} = B^{-1}C^{-1} \qquad \ldots(i)$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

and $\quad \text{Adj } B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$

$\therefore \quad B^{-1} = \frac{1}{|B|}\text{ Adj } B = -\frac{1}{1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ ...(*ii*)

$$C = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 4 & 7 \\ 3 & 5 \end{vmatrix} = 20 - 21 = -1$$

$$\text{Adj } C = \begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix}$$

$\therefore \quad C^{-1} = \frac{1}{|C|}\text{ Adj } C = -\frac{1}{1}\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix}$ ...(*iii*)

From (*i*), (*ii*) and (*iii*), we have

$$A = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 15+6 & -21-8 \\ -10-3 & 14+4 \end{bmatrix} = \begin{bmatrix} 21 & -29 \\ -13 & 18 \end{bmatrix}$$

15. Consider $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$\therefore \quad \frac{dx}{d\theta} = 3a\cos^2\theta\cdot(-\sin\theta),\ \frac{dy}{d\theta} = 3a\sin^2\theta\cdot\cos\theta = -3a\sin\theta\cos^2\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan\theta) = -\sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\sec^2\theta \times \left(\frac{1}{-3a\cos^2\theta\sin\theta}\right) = \frac{1}{3a}\sec^4\theta\,\text{cosec}\,\theta$$

$$\left.\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{4}} = \frac{1}{3a}\sec^4\frac{\pi}{4}\cdot\text{cosec}\,\frac{\pi}{4} = \frac{1}{3a}\times(\sqrt{2})^4\cdot\sqrt{2} = \frac{4\sqrt{2}}{3a}.$$

OR

Consider $\quad y = \log\left(\frac{x}{a+bx}\right)^x = x\log\left(\frac{x}{a+bx}\right) = x[\log x - \log(a+bx)]$

$$\frac{dy}{dx} = x\cdot\left[\frac{1}{x} - \frac{b}{a+bx}\right] + \log\left(\frac{x}{a+bx}\right)\times 1 = x\left[\frac{a+bx-bx}{x(a+bx)}\right] + \log\left(\frac{x}{a+bx}\right)$$

$$\frac{dy}{dx} = \frac{a}{a+bx} + \frac{y}{x}$$

$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx}$ ...(*i*)

Differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax\cdot b}{(a+bx)^2}$$

$$x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow \qquad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{ax}{a+bx}\right)^2$$

$$\Rightarrow \qquad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2 \qquad \text{[From } (i)\text{]}$$

Hence, $x^3\left(\frac{d^2y}{dx^2}\right) = \left(x\frac{dy}{dx} - y\right)^2$

16. $$\int \frac{\log x}{(1+\log x)^2}dx = \int \frac{t}{(1+t)^2}\cdot e^t\,dt$$

Let $\log x = t$
$\Rightarrow \quad x = e^t$
$\Rightarrow \quad dx = e^t\,dt$

$$= \int e^t\left\{\frac{1}{1+t} - \frac{1}{(1+t)^2}\right\}dt$$

$$= e^t\cdot\frac{1}{1+t} + C = e^{\log x}\cdot\frac{1}{1+\log x} + C = \frac{x}{1+\log x} + C$$

$\left[\text{Using} \int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C\right]$

OR

Consider $\int \frac{\sqrt{x^2+1}\,[\log(x^2+1) - 2\log x]}{x^4}dx$

$$= \int \frac{\sqrt{x^2+1}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]}{x^4}dx$$

$$= \int \frac{x\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)}{x^4}dx$$

Let $1+\frac{1}{x^2} = t$

$\Rightarrow -\frac{2}{x^3}dx = dt$

$$= -\frac{1}{2}\int \underset{②}{\sqrt{t}}\,\underset{①}{\log t}\,dt = -\frac{1}{2}\left[\log t\cdot\frac{2}{3}t^{\frac{3}{2}} - \int \frac{1}{t}\cdot\frac{2}{3}\cdot t^{\frac{3}{2}}\,dt\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{1}{3}\int t^{\frac{1}{2}}\,dt = -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\log\left(1+\frac{1}{x^2}\right) + \frac{2}{9}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}} + C$$

17. Here, $a = 2$, $b = 3$, $f(x) = 3x^2 - 2x$, $h = \frac{3-2}{n} \Rightarrow nh = 1$ ...(*i*)

$\int_2^3 (3x^2 - 2x)dx = \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h)] + \ldots + f(2+\overline{n-1}h)$ ...(*ii*)

$f(2) = 3(2)^2 - 2\times 2 = 12 - 4 = 8$

$f(2+h) = 3(2+h)^2 - 2(2+h) = 3(4+h^2+4h) - 4 - 2h = 3h^2 + 10h + 8$

$f(2+2h) = 3(2+2h)^2 - 2(2+2h) = 3(4+4h^2+8h) - 4 - 4h = 12h^2 + 20h + 8$

$f(2+\overline{n-1}h) = 3\{2+(\overline{n-1})h\}^2 - 2\{2+(\overline{n-1})h\} = 3\{4+(n-1)^2h^2+4(n-1)h\} - 4 - 2(n-1)h$

$= 3(n-1)^2h^2 + 10(n-1)h + 8$

Substituting in (*ii*), we get

$$= \lim_{h \to 0} h[(8) + (3h^2 + 10h + 8) + (12h^2 + 20h + 8) + ... + \{3(n-1)^2h^2 + 10(n-1)h + 8\}]$$

$$= \lim_{h \to 0} h[3h^2\{1 + 4 + ... + (n-1)^2\} + 10h\{1 + 2 + ... (n-1)\} + 8n]$$

$$= \lim_{h \to 0} h\left[3h^2 \cdot \frac{(n-1)n(2n-1)}{6} + 10h\frac{(n-1)n}{2} + 8n\right]$$

$$= \lim_{h \to 0}\left[\frac{(nh-h)(nh)(2nh-h)}{2} + 5(nh-h)(nh) + 8nh\right]$$

$$= \lim_{h \to 0}\left[\frac{(1-h)(1)(2-h)}{2} + 5(1-h)(1) + 8 \times 1\right] \qquad \text{[From } (i)]$$

$$= \frac{1 \times 1 \times 2}{2} + 5 \times 1 \times 1 + 8 = 1 + 5 + 8 = 14$$

18. Consider $x^2dy = (2xy + y^2)dx$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2xy + y^2}{x^2} \qquad ...(i)$$

Let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

From (*i*), $\qquad v + x\dfrac{dv}{dx} = \dfrac{2vx^2 + v^2x^2}{x^2} = 2v + v^2$

$$\Rightarrow \qquad x\frac{dv}{dx} = v^2 + v$$

$$\Rightarrow \qquad \int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \frac{1}{2 \times \frac{1}{2}}\log\left|\frac{v + \frac{1}{2} - \frac{1}{2}}{v + \frac{1}{2} + \frac{1}{2}}\right| = \log|x| + \log C$$

$$\Rightarrow \qquad \log\left|\frac{v}{v+1}\right| = \log|Cx|$$

$$\Rightarrow \qquad \frac{v}{v+1} = Cx \Rightarrow \frac{\frac{y}{x}}{\frac{y}{x} + 1} = Cx$$

$$\Rightarrow \qquad \frac{y}{y+x} = Cx \qquad ...(ii)$$

Given $y = 1$ when $x = 1$

$$\Rightarrow \qquad \frac{1}{2} = C$$

From (*ii*), $\qquad \dfrac{y}{y+x} = \dfrac{1}{2}x$

$\Rightarrow \qquad 2y = xy + x^2$ is particular solution.

19. Consider equation $\frac{dy}{dx} - 2y = \cos 3x$

Here, $P(x) = -2$, $Q(x) = \cos 3x$

$$\text{Integrating factor } (I.F.) = e^{-2\int 1 \cdot dx} = e^{-2x}$$

Solution is $\quad (I.F.)\, y = \int \{(I.F.)Q(x)\}dx$

$$e^{-2x} \cdot y = \int e^{-2x} \cos 3x \, dx \qquad ...(i)$$

Consider $\quad I = \int \underset{①}{e^{-2x}} \underset{②}{\cos} 3x \, dx$

$$= e^{-2x} \cdot \frac{\sin 3x}{3} - \int -2e^{-2x} \cdot \frac{\sin 3x}{3} dx$$

$$= \frac{1}{3} e^{-2x} \sin 3x + \frac{2}{3}\left[e^{-2x} \cdot \left(-\frac{\cos 3x}{3}\right) - \int -2e^{-2x} \cdot \left(-\frac{\cos 3x}{3}\right) dx \right]$$

$$I = \frac{1}{3} e^{-2x} \sin 3x - \frac{2}{9} e^{-2x} \cos 3x - \frac{4}{9} I$$

$$\Rightarrow \quad \frac{13}{9} I = \frac{1}{9} e^{-2x} (3 \sin 3x - 2 \cos 3x)$$

$$\Rightarrow \quad I = \frac{1}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x)$$

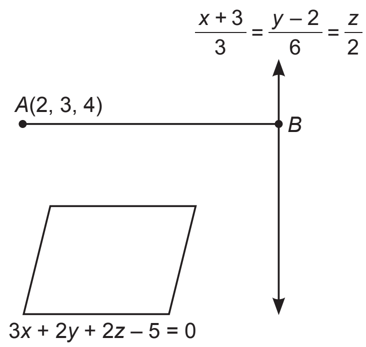
Substituting in (i), we get

$$e^{-2x} \cdot y = \frac{1}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x) + C$$

$$\Rightarrow \quad y = \frac{1}{13} (3 \sin 3x - 2 \cos 3x) + Ce^{2x} \text{ is required solution.}$$

20. General point on the line

$$\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2} = k(\text{say}) \text{ is}$$



$B(3k - 3, 6k + 2, 2k) \qquad ...(i)$

DR's of AB:

$3k - 3 - 2,\ 6k + 2 - 3,\ 2k - 4$

i.e., $3k - 5,\ 6k - 1,\ 2k - 4$

If AB is parallel to the plane $3x + 2y + 2z - 5 = 0$

then $3(3k - 5) + 2(6k - 1) + 2(2k - 4) = 0$

$\Rightarrow \quad 9k - 15 + 12k - 2 + 4k - 8 = 0$

$\Rightarrow \quad 25k = 25 \Rightarrow k = 1$

From (i) coordinates of point B are $(3 - 3, 6 + 2, 2)$, i.e. $B(0, 8, 2)$

$\therefore \quad$ Distance $AB = \sqrt{(0-2)^2 + (8-3)^2 + (2-4)^2} = \sqrt{4 + 25 + 4} = \sqrt{33}$ units

21. General equation of a plane through the points (1, 0, 0) is

$$a(x-1) + b(y-0) + c(z-0) = 0$$

$$\Rightarrow \quad a(x-1) + by + cz = 0 \qquad ...(i)$$

This plane passes through the point (0, 1, 0)

$$\Rightarrow \quad -a + b + 0 = 0 \quad \Rightarrow \quad a = b \qquad ...(ii)$$

and plane (i) makes angle $\frac{\pi}{4}$ with plane $x + y = 3$

$$\therefore \quad \frac{a \cdot 1 + b \cdot 1 + c \cdot 0}{\sqrt{a^2+b^2+c^2}\,\sqrt{1+1+0}} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{a+b}{\sqrt{a^2+b^2+c^2}} = 1 \Rightarrow (a+b)^2 = a^2 + b^2 + c^2$$

$$\Rightarrow \quad a^2 + b^2 + 2ab = a^2 + b^2 + c^2$$

$$\Rightarrow \quad c^2 = 2ab \quad \Rightarrow \quad c^2 = 2a^2 \qquad \text{[from } (ii)]$$

$$\Rightarrow \quad c = \pm\sqrt{2}\,a \qquad ...(iii)$$

Substituting in (i) from (ii) and (iii), we get

Plane is $a(x-1) + ay \pm \sqrt{2}az = 0 \quad \Rightarrow \quad x - 1 + y \pm \sqrt{2}z = 0 \quad \Rightarrow \quad x + y \pm \sqrt{2}z - 1 = 0$

DR's of normal to the plane are 1, 1, $\pm\sqrt{2}$

22. A: even number ; B: odd number

$$A : B = 2 : 1$$

$$\therefore \quad P(A) = \frac{2}{3},\ P(B) = \frac{1}{3}$$

$$p = \frac{2}{3},\ q = \frac{1}{3},\ n = 3$$

Probability of r, successes is

$$P(r) = {}^3C_r\left(\frac{1}{3}\right)^{3-r}\left(\frac{2}{3}\right)^r \quad ; \quad r = 0, 1, 2, 3$$

$$P(0) = {}^3C_0\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 = \frac{1}{27} \quad ; \quad P(1) = {}^3C_1\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{6}{27}$$

$$P(2) = {}^3C_2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{12}{27} \quad ; \quad P(3) = {}^3C_3\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Table for probability distribution and calculation of mean is as follows:

X	$P(X)$	$X \cdot P(X)$
0	$\frac{1}{27}$	0
1	$\frac{6}{27}$	$\frac{6}{27}$
2	$\frac{12}{27}$	$\frac{24}{27}$
3	$\frac{8}{27}$	$\frac{24}{27}$
	$\sum P(X) = \frac{27}{27} = 1$	$\sum X \cdot P(X) = \frac{54}{27} = 2$

Mean = $\Sigma X \cdot P(X) = 2$

23. S : getting a six

$$P(S) = \frac{1}{6}, P(\overline{S}) = \frac{5}{6}$$

$$P(A) = P(1) + P(3) + P(5) + ...$$

$$= P(S) + [P(\overline{S})]^2 P(S) + [P(\overline{S})]^4 P(S) + ...$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + ... = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

$$P(B) = P(2) + P(4) + P(6) + ...$$

$$= P(\overline{S})\ P(S) + [P(\overline{S})]^3 P(S) + [P(\overline{S})^5]\ P(S) + ...$$

$$= \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + ... = \frac{\frac{5}{36}}{1 - \frac{25}{36}} = \frac{5}{11}$$

Decision of referee to ask captain of team A to start first is not fair as whosoever starts first will have more chances of winning.

24. Let $$P(n) : A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$P(1) : A = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A, \quad \text{true}$$

Let $P(k)$ be true, i.e.

$$A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \qquad ...(i)$$

To show $P(k + 1)$ is true, i.e.

$$A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix} = \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

Consider $$A^{k+1} = A^k A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \qquad \text{[from } (i)]$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$\therefore$ $P(k)$ true $\Rightarrow P(k + 1)$ is true, $P(1)$ is also true, so statement is true for all positive integers by the principle of mathematical induction.

OR

Consider, $$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 0 \\ ax+a^2 & a & -1 \\ ax^2+a^2x & ax & a \end{vmatrix} \qquad [\text{Performing } C_1 \to C_1 + aC_2]$$

$$f(x) = 1[a^2x + a^3 + ax^2 + a^2x] = ax^2 + 2a^2x + a^3$$

$\therefore \quad f(2x) - f(x) = [a(2x)^2 + 2a^2(2x) + a^3] - [ax^2 + 2a^2x + a^3]$

$= 4ax^2 + 4a^2x + a^3 - ax^2 - 2a^2x - a^3 = 3ax^2 + 2a^2x$

25. Given $A = \{x \in W : 0 \le x \le 12\}$

and $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

For reflexive: Let for $a \in A$

$(a, b) \in R \Rightarrow |a - a|$ is a multiple of 4

$\Rightarrow$ 0 is a multiple of 4, true.

Hence, reflexive.

For symmetric: Let for $a, b \in A$

$(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4

$\Rightarrow |-(b - a)|$ is a multiple of 4

$\Rightarrow |b - a|$ is a multiple of 4

$\therefore (a, b) \in R \Rightarrow (b - a) \in R$. Hence, symmetric.

For transitive: Let for $a, b, c \in A$

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow |a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4

$\Rightarrow (a - b)$ is a multiple of 4 and $(b - c)$ is a multiple of 4

$\Rightarrow (a - b) + (b - c) = a - c$ is a multiple of 4

$\Rightarrow |a - c|$ is a multiple of 4

$\Rightarrow (a, b) \in R$

$\therefore (a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$. Hence, transitive.

As relation R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

Set of elements related to 2 $\Rightarrow (a, 2) \in R$

$\Rightarrow |a - 2|$ is a multiple of 4

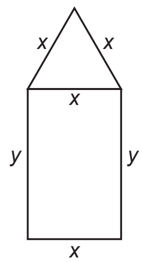
$\Rightarrow a - 2 = 4\lambda \Rightarrow a = 2 + 4\lambda$

$\Rightarrow a = 2, 6, 10 \in A$

$\therefore [2] = \{2, 6, 10\}$

26. Let length and breadth of the rectangle be y m and x m respectively. An equilateral triangle is along the breadth.

Perimeter of window = 12 m



$\therefore \quad 3x + 2y = 12 \Rightarrow y = \dfrac{12 - 3x}{2}$ ...(*i*)

Area of window, $(A) = xy + \dfrac{\sqrt{3}}{4}x^2$

$= x\left[\dfrac{12 - 3x}{2}\right] + \dfrac{\sqrt{3}}{4}x^2$ [from (*i*)]

$A = 6x - \dfrac{3}{2}x^2 + \dfrac{\sqrt{3}}{4}x^2$

On differentiating both sides, w.r.t. x, we get

$$\frac{dA}{dx} = 6 - 3x + \frac{\sqrt{3}}{2}x$$

For maximum or minimum area,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2} - 3\right)x = -6$$

$$\Rightarrow x = \frac{6}{3 - \frac{\sqrt{3}}{2}} = \frac{12}{6 - \sqrt{3}} \text{m}$$

$$\frac{d^2A}{dx^2} = \left(\frac{\sqrt{3}}{2} - 3\right)$$

$$\left.\frac{d^2A}{dx^2}\right]_{x = \frac{12}{6-\sqrt{3}}} < 0$$

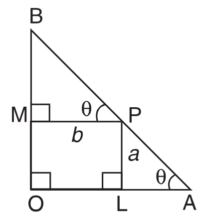
$\therefore$ for $x = \dfrac{12}{6 - \sqrt{3}}$, area is maximum.

Substituting in (i), we get

$$y = 6 - \frac{3}{2}\left(\frac{12}{6 - \sqrt{3}}\right) = \frac{36 - 6\sqrt{3} - 18}{6 - \sqrt{3}} = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}} \text{ m}$$

Hence, for breadth $= \dfrac{12}{6 - \sqrt{3}}$ m and length $= \dfrac{18 - 6\sqrt{3}}{6 - \sqrt{3}}$ m, area is maximum.

OR



Hypotenuse (h) = $AB = AP + PB$ ...(i)

Now $\dfrac{AP}{a} = \text{cosec}\,\theta \Rightarrow AP = a\,\text{cosec}\,\theta$

and $\dfrac{BP}{b} = \sec\theta \Rightarrow BP = b\sec\theta$

From (i), $h = a\,\text{cosec}\,\theta + b\sec\theta$ ...(ii)

$$\frac{dh}{d\theta} = a(-\text{cosec}\,\theta\cot\theta) + b(\sec\theta\tan\theta)$$

For minimum h, $\dfrac{dh}{d\theta} = 0 \Rightarrow a\,\text{cosec}\,\theta\cot\theta = b\sec\theta\tan\theta$

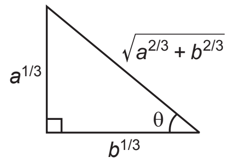
$$\Rightarrow \quad \frac{a}{b} = \tan^3\theta \quad \Rightarrow \quad \tan\theta = \left(\frac{a}{b}\right)^{1/3}$$

$$\frac{d^2h}{d\theta^2} = -a[\text{cosec}\,\theta(-\text{cosec}^2\,\theta) + \cot\theta(-\text{cosec}\,\theta\cot\theta)] + b[\sec\theta\cdot\sec^2\theta + \tan\theta\sec\theta\tan\theta]$$

$\dfrac{d^2h}{d\theta^2} > 0$ for $\tan\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$

$\therefore h$ is minimum for $\tan\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$

$\Rightarrow \qquad \sin\theta = \dfrac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}, \cos\theta = \dfrac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$



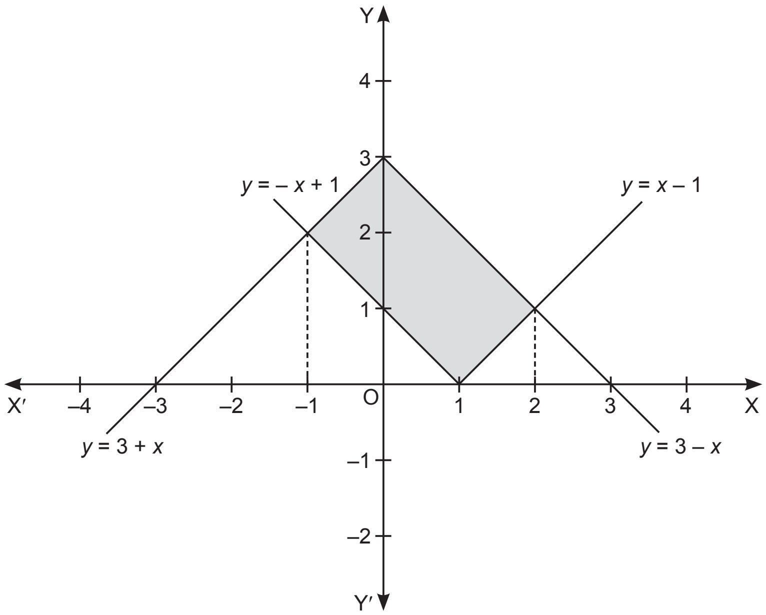
Putting in (*ii*), we get $\quad h = a\cdot\dfrac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + b\cdot\dfrac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$

$= (a^{2/3} + b^{2/3})^{3/2}$

27. Given curves: $\quad y = |x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases}$

and $\quad y = 3 - |x| = \begin{cases} 3-x, & x \geq 0 \\ 3+x, & x < 0 \end{cases}$

Plotting the graph of the curves we notice we have to find the shaded area.



Point of intersection of $y = -x + 1$ and $y = x + 3$ is $x + 3 = -x + 1 \Rightarrow 2x = -2 \Rightarrow x = -1$

and for $\quad y = 3 - x$ and $y = x - 1$

$3 - x = x - 1 \;\Rightarrow\; 4 = 2x \;\Rightarrow\; x = 2$

$$\text{Area} = \int_{-1}^{0}(3+x)dx + \int_{0}^{2}(3-x)dx - \int_{-1}^{1}(-x+1)dx - \int_{1}^{2}(x-1)dx$$

$$= \left[3x + \frac{x^2}{2}\right]_{-1}^{0} + \left[3x - \frac{x^2}{2}\right]_{0}^{2} - \left[\frac{-x^2}{2} + x\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= 0 - \left(-3 + \frac{1}{2}\right) + (6-2) - 0 - \left(-\frac{1}{2} + 1\right) + \left(-\frac{1}{2} - 1\right) - (2-2) + \left(\frac{1}{2} - 1\right)$$

$$= \frac{5}{2} + 4 - \frac{1}{2} - \frac{3}{2} - \frac{1}{2}$$

$= 4$ sq units

28. Let x skilled helpers and y unskilled helpers are employed.

Then *LPP* is

Minimise $Z = 225x + 200y$

subject to the constraints

$$x \geq 0, y \geq 0$$

$$x + y \leq 10$$

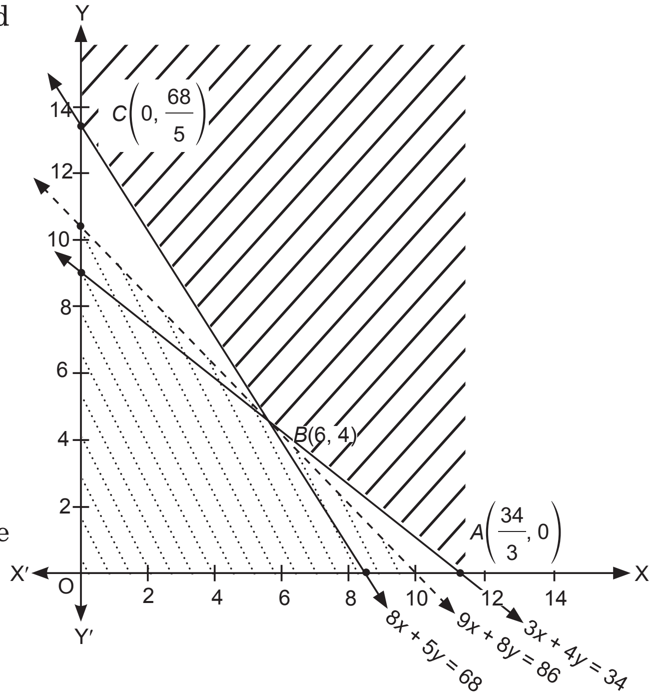
$$300x + 400y \geq 3400$$

$\Rightarrow$ $$3x + 4y \geq 34$$

$$80x + 50y \geq 680$$

$\Rightarrow$ $$8x + 5y \geq 68$$

On plotting the inequations, we notice shaded portion is feasible solution.



Possible for minimum Z are

$A\left(\frac{34}{3}, 0\right)$, $B(6, 4)$, $C\left(0, \frac{68}{5}\right)$

Points	**Z = 225x + 200y**	**Value**
$A\left(\frac{34}{3}, 0\right)$	2550 + 0	2550
$B(6, 4)$	1350 + 800	2150
$C\left(0, \frac{68}{5}\right)$	0 + 2720	2720

← Minimum

Z is minimum at $B(6, 4)$. Since region is unbounded; we draw graph of inequation $225x + 200y < 2150 \Rightarrow 9x + 8y < 86$.

Since graph of inequation $9x + 8y < 86$ does not have any point common with feasible region. So, $B(6, 4)$ represents minimum, i.e. $x = 6, y = 4$.

$\therefore$ 6 skilled workers and 4 unskilled workers must be employed for a minimum cost of ₹ 2150.

29. Given points are $A(4, 1, 2)$; $B(5, x, 6)$; $C(5, 1, -1)$; $D(7, 4, 0)$

If points are coplanar, then $[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = 0$

$$\overrightarrow{AB} = (5-4)\hat{i} + (x-1)\hat{j} + (6-2)\hat{k} = \hat{i} + (x-1)\hat{j} + 4\hat{k}$$

$$\overrightarrow{AC} = (5-4)\hat{i} + (1-1)\hat{j} + (-1-2)\hat{k} = \hat{i} - 3\hat{k}$$

$$\overrightarrow{AD} = (7-4)\hat{i} + (4-1)\hat{j} + (0-2)\hat{k} = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

$$[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = \begin{vmatrix} 1 & x-1 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$\Rightarrow$ $1(9) - (x-1)(7) + 4(3) = 0 \Rightarrow 9 - 7x + 7 + 12 = 0 \Rightarrow 7x = 28 \Rightarrow x = 4$

OR

$$\text{LHS} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 5 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}(6-5) - \hat{j}(-9-10) + \hat{k}(3+4) = \hat{i} + 19\hat{j} + 7\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 1 & 19 & 7 \end{vmatrix} = \hat{i}(-7-38) - \hat{j}(14-2) + \hat{k}(38+1) = -45\hat{i} - 12\hat{j} + 39\hat{k}$$

$$\text{RHS} = (\vec{a} \cdot \vec{c}) = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = 4 - 1 - 6 = -3$$

$$(\vec{a} \cdot \vec{b}) = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 5\hat{k}) = 6 + 2 + 10 = 18$$

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = -3(3\hat{i} - 2\hat{j} + 5\hat{k}) - 18(2\hat{i} + \hat{j} - 3\hat{k})$$

$$= -9\hat{i} + 6\hat{j} - 15\hat{k} - 36\hat{i} - 18\hat{j} + 54\hat{k}$$

$$= -45\hat{i} - 12\hat{j} + 39\hat{k}$$

Hence, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$